Masters Presentation

D. Dakota Blair

March 20, 2007

Abstract

This is a short introduction to a few aspects of simplicial topology which coupled with the correct definitions extends the Matrix Tree Theorem to acyclic complexes.

Definition 1 (Spanning Tree) *Given a graph G with $|G| = n$, a spanning tree of G is a set T of edges and verticies such that every vertex of G is in T, T has no cycles ($\tilde{H}_1 = 0$) and the number of edges of T is $\|T\| = n - 1$.*

Definition 2 (Incidence (boundary) matrix of $G, \partial(G)$) *Given a graph G with e edges and v verticies, define an $e \times v$ matrix $\partial(G)$ with $\partial(G)_{e_0,v_0} = 1$ if e_0 is incident with v_0 and 0 otherwise.*

Definition 3 (The Laplacian matrix of G, L(G)) $L(G) = \partial(G)\partial(G)^T$

Definition 4 (The reduced Laplacian matrix of G, $L_r(G)$) *The Laplacian matrix of G the row of a single vertex removed.*

Theorem 1 (Matrix Tree Theorem) *A graph G has $|\det L_r(G)|$ spanning trees.*

Proof:

$$\det L_r(G) = \det \partial_r(G)\partial_r(G)^T = \sum_T (\det \partial_r(T))^2 = \sum_T (\pm 1)^2$$

where the last two sums are over every spanning tree of G. Q.E.D.

Definition 5 (n-simplex) *The n-dimensional triangle, formally*

$$\Delta^n = \{(t_1, \ldots, t_{n+1}) \in \mathbb{R}^n | \, t_1 + \cdots + t_n = 1 \text{ and } t_i \geq 0\}$$

Definition 6 (The boundary homomorphism ∂) *Define $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by specifying its value on basis elements, that is,*

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{\{v_0, \ldots v_{i-1}, v_{i+1}, \ldots, v_n\}}$$

where $\Delta_n(X)$ is the free abelian group with basis the open n-simplicies of X.

Theorem 2 $\partial^2 = 0$

Proof: Consider the sequence

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

From the definition we have

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{\{v_0, \ldots v_{i-1}, v_{i+1}, \ldots, v_n\}}$$

therefore

$$\begin{aligned} \partial_{n-1}\partial_n(\sigma) &= \sum_{j<i} (-1)^i (-1)^j \sigma|_{\{v_0, \ldots v_{j-1}, v_{j+1} \ldots v_{i-1}, v_{i+1}, \ldots, v_n\}} \\ &\quad + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{\{v_0, \ldots v_{i-1}, v_{i+1} \ldots v_{j-1}, v_{j+1}, \ldots, v_n\}} = 0 \end{aligned}$$

The two sums are symmetric in i and j, we may replace i with j in the latter to see that the whole sum is zero. Q.E.D.

Definition 7 (Simplicial Complex) *A set K of simplicies which satisfies*

- *any face of a simplex in K is also in K.*
- *for any two simplicies $s_1, s_2 \in K$, their intersection is a face of both s_1 and s_2.*

Note that the empty set is $\partial\Delta^0$ and hence is a face of every complex.

Definition 8 (The nth Simplicial Homology Group $H_n(X)$) *In the chain complex*

$$\cdots \longrightarrow \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X) \longrightarrow \cdots \longrightarrow \delta_0 \xrightarrow{\partial_0} 0$$

define $H_n(x)$ as

$$H_n(X) = Ker\,\partial_n / Im\,\partial_{n+1}$$

Definition 9 (The Reduced Homology Group $\tilde{H}_n(X)$) *Given a nonempty X, in the chain complex*

$$\cdots \longrightarrow \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X) \longrightarrow \cdots \longrightarrow \delta_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

define $\tilde{H}_n(x)$ as

$$\tilde{H}_n(X) = Ker\,\partial_n / Im\,\partial_{n+1}.$$

Here $\varepsilon(\sum_i n_i\sigma_i) = \sum_i n_i$.

Definition 10 (Acyclic Complex) *A simplicial complex K which is connected and $H_i(K) = 0$ for all $i > 0$.*

Definition 11 (Simplicial Spanning Tree) *Given a k-dimensional complex* Δ *a simplical panning tree is a k-dimensional complex* T *containing all* $k-1$*-dimensional faces of* Δ *(that is,* $T^{k-1} = \Delta^{k-1}$*) and*

- $\tilde{H}_k = 0$ *(Acyclic)*
- $\tilde{H}_{k-1}$ *is a finite group.*

Theorem 3 (Simplicial Matrix Tree Theorem) *Given a set of facets* U *of a* $k-1$*-spanning tree of* Δ *and reducing* L *by all of* U*, defining* $\Delta_U = U \cup \Delta^{k-2}$ *we may now calculate*

$$\sum_{T \in SST(\Delta)} |\tilde{H}_{k-1}(T)|^2 = \frac{|\tilde{H}_{k-2}(\Delta)|^2}{|\tilde{H}_{k-2}(\Delta_U)|^2} \det L_r.$$