A RELATION ON PARTIAL WORDS

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1. Introduction

Let $\mathbb{N}$ denote the natural numbers, and associate an element $n \in \mathbb{N}$ with its corresponding ordinal number. A *partial function* from X to Y is a function $f : X' \to Y$ such that $X' \subset X$ is the *domain* of f. Let the set of partial functions from X to Y be written as $Y^{(X)}$. Denote the domain of $f \in Y^{(X)}$ as $D(f)$ and the *set of holes of* f as $H(f) = X \setminus D(f)$. Given $n, m \in \mathbb{N}$ let $k = n \pmod m$ be modular reduction of n modulo m, that is, $0 \leq k < m$ and $k \equiv n \pmod m$.

Definition 1.1 (Partial Words)**.** *A **partial word** u of length $|u| = n$ on an alphabet A is an element of $A^{(n)}$.*

With this definition, $A^* = \cup_{n\in\mathbb{N}} A^{(n)}$ forms a monoid under concatenation with identity element ϵ, the empty word.

Example 1.1. *The partial word $c\Diamond b$ is an element of $\{a, b, c\}^{(3)}$.*

Definition 1.2 (Containment)**.** *A partial word u is said to be **contained** in v, written $u \subset v$, if $|u| = |v|, D(u) \subset D(v)$ and $u(i) = v(i)$ for all $i \in D(u)$.*

Example 1.2. *The partial word $c\Diamond b$ is contained in both cab and cub.*

Definition 1.3 (Compatibility)**.** *A partial word u is **compatible** with v, written $u \uparrow v$, if there exists a partial word w such that $u \subset w, v \subset w$.*

Example 1.3. *Extending the previous example, $c\Diamond b \uparrow cab$ and $c\Diamond b \uparrow cub$, but $cab \not\uparrow cub$. Therefore the $\uparrow$ relation is reflexive and symmetric, but not transitive. Another example is $a\Diamond bbc \uparrow aab\Diamond c$.*

Definition 1.4 (Periodic)**.** *A partial word u is p-**periodic** if $i \equiv j \pmod p$ and $i, j \in D(u)$ implies $u(i) = u(j)$.*

Example 1.4. *The word abcabc is 3-periodic, and the partial word $abab\Diamond b$ is 2-periodic.*

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Definition 1.5 (Residual Partial Words). *For a partial word u and $i, p \in \mathbb{N}$, let i' be $i' = i \pmod p$ and $k \in \mathbb{N}$ be the greatest integer such that $i' + kp < |u|$. Then the* ***ith residual partial word modulo p of u***, *denoted $u\begin{bmatrix} i \\ p \end{bmatrix}$, is defined to be*

$$u\begin{bmatrix} i \\ p \end{bmatrix} = \prod_{j=0}^{k} u(i' + jp) = u(i')u(i' + p)u(i' + 2p) \cdots u(i' + kp).$$

Example 1.5. *Let $u = (prime)^5$. Then $u\begin{bmatrix} 0 \\ 6 \end{bmatrix} = prime$. If $v = (a\Diamond b)^3$ then $v\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \Diamond\Diamond\Diamond$.*

2. The relation $u^m \uparrow v^n$ on partial words

Lemma 2.1. *Let u and v be partial words, such that for $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$ $u^m \uparrow v^n$ and $p = |u|/n = |v|/m$. If there exists an i such that $0 \leq i < p$ and $u\begin{bmatrix} i \\ p \end{bmatrix}$ is not 1-periodic then $D\left(v\begin{bmatrix} i \\ p \end{bmatrix}\right)$ is empty.*

Lemma 2.2. *If u is a partial word, $i, m, p \in \mathbb{N}$ and $0 \leq i \leq p$ then*

$$\begin{aligned} u^m\begin{bmatrix} i \\ p \end{bmatrix} &= u\begin{bmatrix} i \\ p \end{bmatrix} u\begin{bmatrix} (i - |u|) \\ p \end{bmatrix} u\begin{bmatrix} (i - 2|u|) \\ p \end{bmatrix} \cdots u\begin{bmatrix} (i - (m-1)|u|) \\ p \end{bmatrix} \\ &= \prod_{j=0}^{m-1} u\begin{bmatrix} i - j|u| \\ p \end{bmatrix} \end{aligned}$$

Definition 2.1 (Good Pair). *A pair of partial words (u, v) is a* ***good pair*** *if for all $i \in H(u)$ the word $v^n\begin{bmatrix} i \\ |u| \end{bmatrix}$ is 1-periodic and for all $i \in H(v)$ the word $u^m\begin{bmatrix} i \\ |v| \end{bmatrix}$ is 1-periodic.*

Example 2.3. *The partial words $a\Diamond b$ and $acbadb$ do not form a good pair.*

Theorem 2.4. *If (u, v) is a good pair of partial words, m and n are positive integers such that $u^m \uparrow v^n$ with $\gcd(m, n) = 1$ then there exists a partial word z such that $u \subset z^n$ and $v \subset z^m$.*

Proof. Since $\gcd(m, n) = 1$, there exists p such that $|u|/n = |v|/m = p$. Now assume there is an integer i such that $0 \leq i < p$ and $u\begin{bmatrix} i \\ p \end{bmatrix}$ is not 1-periodic. By Lemma 2.1, $i + jp \pmod{|v|} \in H(v)$ for all j such that $0 \leq j < m$. Then the assumption that (u, v) is a good pair implies $w = u^m\begin{bmatrix} i + jp \\ |v| \end{bmatrix}$ is 1-periodic for all j, and therefore $w \subset a^n$ for some letter a. Note also that $|u| = np$ implies $|w| = |u| = n$. To show $u\begin{bmatrix} i \\ p \end{bmatrix}$ is a permutation of w will be sufficient to complete the proof. By Lemma 2.2

$$w = u^m\begin{bmatrix} i+jp \\ mp \end{bmatrix} = \prod_{k=0}^{m-1} u\begin{bmatrix} i+(j-k|u|)p \\ mp \end{bmatrix}$$
$$= u\begin{bmatrix} i+jp \\ mp \end{bmatrix} u\begin{bmatrix} i+(j-|u|)p \\ mp \end{bmatrix} u\begin{bmatrix} i+(j-2|u|)p \\ mp \end{bmatrix} \cdots u\begin{bmatrix} i+(j-(m-1)|u|)p \\ mp \end{bmatrix}.$$

Note that $i + jp - k|u| = i + (j - kn)p$ and since $\gcd(m, n) = 1$

$$\{j - kn (\bmod m) \big| 0 \leq k < m\} = \{0, 1, \ldots, m-1\}.$$

Therefore define $j_k = j - kn \pmod m$, so

$$w = u^m\begin{bmatrix} i+jp \\ mp \end{bmatrix} = \prod_{k=0}^{m-1} u\begin{bmatrix} i+j_kp \\ mp \end{bmatrix} = u\begin{bmatrix} i+j_0p \\ mp \end{bmatrix} u\begin{bmatrix} i+j_1p \\ mp \end{bmatrix} \cdots u\begin{bmatrix} i+j_{m-1}p \\ mp \end{bmatrix}.$$

Since w is 1-periodic, there is a letter a such that for $0 \leq k < m$ and some $m_k \in \mathbb{N}$

$$u\begin{bmatrix} i+j_kp \\ mp \end{bmatrix} \subset a^{m_k}.$$

Now observe that

$$\left\{k + lm \Big| 0 \leq k < m, 0 \leq l < n - k/m - i/mp\right\} = \{0, 1, \ldots, n-1\}$$

where the condition on l comes from the requirement that the sum $i + kp + lmp$ be less than $|u| = np$. Therefore the partial word w is a permutation of $u\begin{bmatrix} i \\ p \end{bmatrix}$. That is, if $\sigma \in S_n$, the symmetric group on n,

$$w = u^m\begin{bmatrix} i+jp \\ mp \end{bmatrix} = \prod_{k=0}^{m-1} u\begin{bmatrix} i+j_kp \\ mp \end{bmatrix} = \prod_{h=0}^{n-1} u(i + \sigma(h)p) \subset a^n.$$

But w is 1-periodic, and therefore so is $u\begin{bmatrix} i \\ p \end{bmatrix}$ which contradicts our choice of i. Then $u\begin{bmatrix} i \\ p \end{bmatrix}$ is 1-periodic for all i, so u is p-periodic. Thus there is a partial word z_u of length p such that $u \subset z_u^n$. Repeating the argument for v yields a partial word z_v of length p such that $v \subset z_v^m$. Then $u^m \uparrow y^n$ implies that $z_u^{mn} \uparrow z_v^{mn}$ and therefore $z_u \uparrow z_v$. Finally $z = z_u \vee z_y$ is a partial word such that $u \subset z^n$ and $v \subset z^m$. $\square$