THE TWO FIXED CENTERS: AN EXCEPTIONAL INTEGRABLE SYSTEM

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Abstract. It is usually believed that we know everything to be known for any separable Hamiltonian system, i.e. an integrable system in which we can separate the variables in some coordinate system (e.g. see Lichtenberg and Lieberman 1992, *Regular and Chaotic Dynamics*, Springer). However this is not always true, since through the separation the solutions may be found only up to quadratures, a form that might not be particularly useful. A good example is the two-fixed-centers problem. Although its integrability was discovered by Euler in the 18th century, the problem was far from being considered as completely understood. This apparent contradiction stems from the fact that the solutions of the equations of motion in the confocal ellipsoidal coordinates, in which the variables separate, are written in terms of elliptic integrals, so that their properties are not obvious at first sight. In this paper we classify the trajectories according to an exhaustive scheme, comprising both periodic and quasi-periodic ones. We identify the collision orbits (both direct and asymptotic) and find that collision orbits are of complete measure in a 3-D submanifold of the phase space while asymptotically collision orbits are of complete measure in the 4-D phase space. We use a transformation, which regularizes the close approaches and, therefore, enables the numerical integration of collision trajectories (both direct and asymptotic). Finally we give the ratio of oscillation period along the two axes (the 'rotation number') as a function of the two integrals of motion.

Key words: collision orbits, integrable systems, periodic orbits, two-fixed centers problem

1. Introduction – Motivation

Our interest to the two-fixed-centers problem (2FCP) has been motivated by the work of Jakas (1995, 1996) on the interpretation of experiments concerning the acceleration of electrons in atomic collisions of ions with atoms. In these experiments the maximum energy of the emerging electrons is of the order of the energy of the incoming ions. If the acceleration mechanism is a single step process, then due to conservation of momentum and energy the maximum gain in velocity of an electron is $2v_i$, where v_i is the velocity of the

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incoming ion. Since the electron is more than a thousand times less massive than any ion, (for a proton $m_p = 1840m_e$), we see that $E_e \approx E_i$ implies $v_e \gg v_i$ and the acceleration cannot be a one-step process. Jakas (1995) considered the 2FCP as a first approximation for the motion of the electron in the electrostatic field of two ions and suggested that, in this approximation, the electron follows a stable periodic figure-8 orbit around them. In trying to assess the efficiency of this acceleration mechanism, we realized that the periodic orbit suggested by Jakas exists but it is not stable. Moreover we found that all orbits ‘close’ to the periodic one migrate slowly to either of the two centers, where they experience large accelerations and it is, therefore, impossible to follow them numerically. Since we were not able to find a complete solution of the problem in the literature, we decided to initiate a systematic study.

The motion of a point mass, moving in the gravitational field of two fixed attracting centers, is a problem first posed by Euler in the 18th century, as an intermediate step towards the solution of the famous three-body problem. Euler himself, in a series of three papers (Euler, 1766, 1767a, b), was able to integrate the equations of motion for the two-dimensional (2-D) case, i.e. the case where the point mass moves on a plane containing the two attracting centers. Almost a century later Jacobi (1842) showed that the corresponding potential of the full 3-D case is separable in prolate spheroidal coordinates. Another century later Erikson and Hill (1949) found, in explicit form, the third integral of motion of the full three-dimensional (3-D) case (besides the other two ‘classical’ ones, i.e. the total energy and the angular momentum about the axis passing through the two centers). Since then the problem has been considered as a non-exciting example of a separable potential and it is included, as such, in many textbooks of Theoretical Mechanics.

That this is not the case can be understood by the multitude of interesting applications, which appeared in the literature after the paper by Erikson and Hill. Thus, the problem of the two fixed centers has been used, among others, in the calculation of satellite trajectories in the gravitational field of the Earth (Alexeev, 1965; Marchal, 1966, 1986), in the calculation of the energy levels of the positive ion of the hydrogen molecule, H_2^+, (Strand and Reihardt, 1979) and in the acceleration of electrons in atomic collisions (Jakas, 1995, 1996). Generalized forms of the problem have been considered as well, such as the finite dipole (Howard and Wilkinson 1995), the repulsive dipole (Kallrath, 1992) and the equally charged two black holes system (Contopoulos, 1990, 1991).

Although the 2FCP is a separable dynamical system, the qualitative behavior of its solutions was, up to now, not very well understood, probably due to the fact that the solutions are expressed in the form of elliptic functions. One thorough attempt for the classification of the solutions has been done by Deprit (1962), but this publication is not easily available. The exaustive classification by Alexeev (1965) is of limited usefulness as well, since only the abstract is available in English. A similar work, but restricted only to a subclass of the solutions, has been carried out by Strand and Reinhardt (1979). Finally Contopoulos and Papadaki (1993) calculated the initial conditions and the characteristic exponents of many periodic trajectories and completed the classification by Charlier (1902). However the above authors did not construct a phase portrait of the dynamical system.

2. Basic Hamiltonian – Equations of Motion

In the present work we focus our interest on the (simpler to study) 2-D case of the problem, where the trajectory of the third body lies on a plane and we assume, without loss of generality, that the third body has unit mass. Then the Hamiltonian of the dynamical system is written, in cartesian co-ordinates, x–y, as

$$H(x, y, p_x, p_y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} - \frac{\alpha_1}{r_1} - \frac{\alpha_2}{r_2} \equiv E \tag{1}$$

where $\alpha_1 = 2\mu$, $\alpha_2 = 2(1-\mu)$ ($\mu \in [0, 1]$ is the mass parameter), $r_1 = \sqrt{(x+1)^2 + y^2}$ and $r_2 = \sqrt{(x-1)^2 + y^2}$. We note that, in the variables used in Equation (1), the distance between the two centers is equal to $\alpha_1 + \alpha_2 = 2 = -\alpha$. For later use (Equation (7)) we define also the *asymmetry mass parameter*, β, which is equal to $\beta = \alpha_1 - \alpha_2 = 4\mu - 2$. We note however that in the numerical examples we restrict our attention to the case of equal masses, so that $\mu = 0.5$ and, therefore, $\beta = 0$.

Following Euler (1766, 1767a, b) and Jacobi (1842; see also e.g. Charlier, 1902; Thirring, 1977; Strand and Reinhardt, 1979), we write the above Hamiltonian in elliptic–hyperbolic coordinates, through the canonical (point-)transformation:

$$\xi = \frac{r_1 + r_2}{2}, \quad \eta = \frac{r_1 - r_2}{2} \tag{2}$$

Then the 'old' variables, as functions of the 'new', are given by

$$x = \xi\eta \tag{3}$$

$$y = (\operatorname{sign} y)\sqrt{(\xi^2 - 1)(1 - \eta^2)} \tag{4}$$

$$p_x = \frac{\eta(\xi^2 - 1)p_\xi + \xi(1 - \eta^2)p_\eta}{\xi^2 - \eta^2} \tag{5}$$

$$p_y = (\operatorname{sign} y)\frac{\sqrt{(\xi^2-1)(1-\eta^2)}(\xi p_\xi - \eta p_\eta)}{\xi^2-\eta^2}$$
$$= \frac{y}{\xi^2-\eta^2}(\xi p_\xi - \eta p_\eta) \tag{6}$$

The Hamiltonian H, in the new variables $(\xi, \eta, p_\xi, p_\eta)$, becomes

$$H = \frac{1}{\xi^2-\eta^2}\left[\frac{1}{2}(\xi^2-1)^2 p_\xi^2 + \alpha\xi + \frac{1}{2}(1-\eta^2)^2 p_\eta^2 + \beta\eta\right] \tag{7}$$

Finally we change the time scale, by multiplying the Hamiltonian function by $\xi^2-\eta^2$, noting that this quantity is positive everywhere except when the moving body collides with one of the two centers, in which case it is equal to zero. At the same time we switch to the extended phase space, where the additional co-ordinate is time, t, and the corresponding momentum, p_t, is equal to $-E$, where by E we denote the numerical value of the Hamiltonian function (Equation (7)). The new Hamiltonian, $K(\xi, \eta, \tau, p_\xi, p_\eta, p_\tau)$,

$$K = \frac{1}{2}(\xi^2-1)^2 p_\xi^2 + \alpha\xi + p_t\xi^2 + \frac{1}{2}(1-\eta^2)^2 p_\eta^2 + \beta\eta - p_t\eta^2 \tag{8}$$

has in the extended phase space a numerical value equal to zero.

We observe that the Hamiltonian is the sum of two parts, K_ξ and K_η, the first depending only on ξ and p_ξ and the second only on η and p_η. Since the value of the Hamiltonian is by definition zero, the two parts should have opposite values, and satisfy the relation

$$K_\xi \equiv -K_\eta \equiv \gamma \tag{9}$$

In this way we have separated the variables and, at the same time, we have recovered the integral of Erikson and Hill (1949), as expressed by Strand and Reinhardt (1979)

$$G = -\frac{1}{\xi^2-\eta^2}(\eta^2 K_\xi + \xi^2 K_\eta) \tag{10}$$

since simply

$$G \equiv \gamma \tag{11}$$

3. Classification of Orbits

Following Charlier (1902), Deprit (1962) and Strand and Reinhardt (1979), the orbits may be classified, according to the available region of configuration space defined by the values of the two integrals of motion, into three basic classes (see Figure 1). In the first class, $P1$, belong the trajectories that lie within an elliptic annulus encircling the two attracting centers. In the second

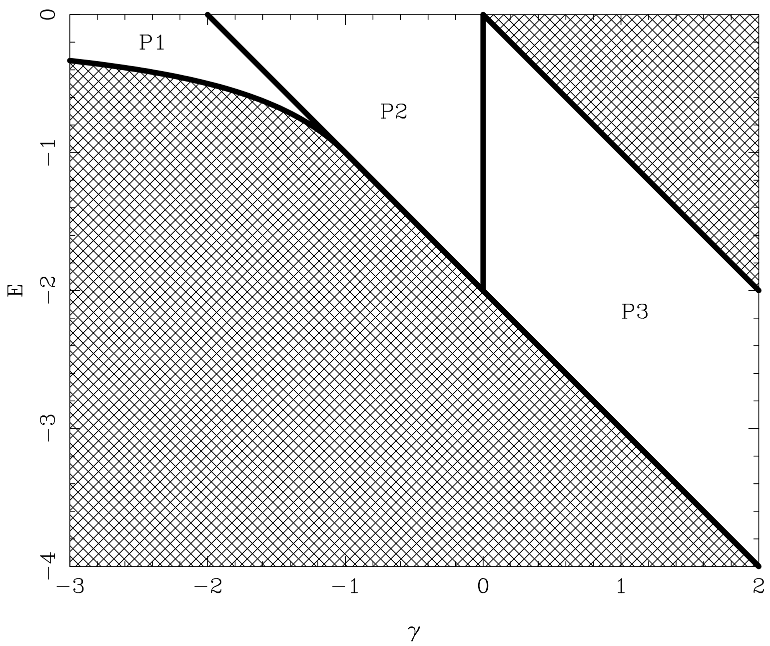

Figure 1. The three main classes of orbits, as defined in the γ–E space.

class, $P2$, belong the trajectories that lie within a simply connected region that contains both attracting centers. Finally in the third class, $P3$, belong the trajectories that lie within two disconnected regions, each one containing one of the attracting centers. The configuration space and the most characteristic periodic orbits are shown in Figure 2. For any value of the energy integral, one unstable isolated periodic orbit of class $P2$ lies on the y-axis ($\eta = 0$) and a pair of stable periodic orbits of class $P1$ lie on an ellipse ($\xi = \text{const.}$), one orbit described clockwise and the other counter-clockwise (e.g. see Broucke, 1980; Meletlidou and Ichtiaroglou, 1999). These orbits may be easily identified on a surface of section, from the hyperbolic and elliptic, correspondingly, structure of the invariant curves in their vicinity (see Figure 3). The figure-8 orbit however does not appear explicitly on the surface of section. The reason is that, as we were able to conjecture from numerical evidence, all periodic orbits, besides those mentioned above, are non-isolated. Therefore the structure of the invariant curves in their vicinity is parabolic and their characteristic exponents are zero. Indications of this general property were found numerically by Contopoulos and Papadaki (1993).

In an integrable dynamical system all trajectories are either periodic or quasi-periodic. However in the case of the two-fixed-centers problem the situation appears more complicated. The reason is that, usually, we tend to consider a dynamical system as integrable if its Hamiltonian can be expressed

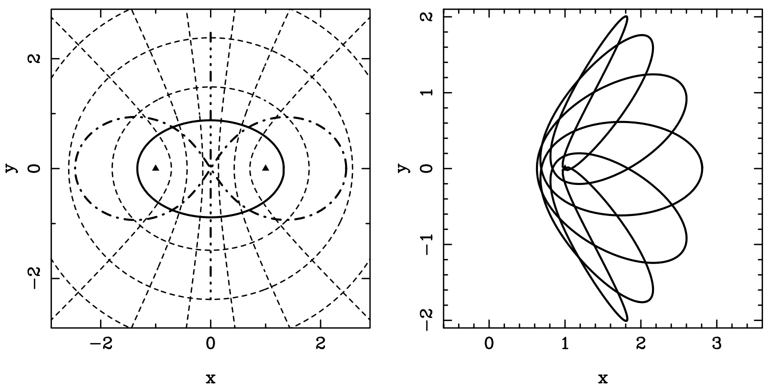

Figure 2. (a) (Left) The figure-8 non-isolated periodic orbit (dot-dashed) and the only isolated ones: unstable (3-dot-dashed line) and stable (solid line) orbits on an elliptic, ξ–η, coordinates grid. (b) (Right) A non-periodic trajectory of class *P*3. It fills densely the available configuration space defined by the zero velocity curve and, therefore, approaches arbitrarily close the attracting center.

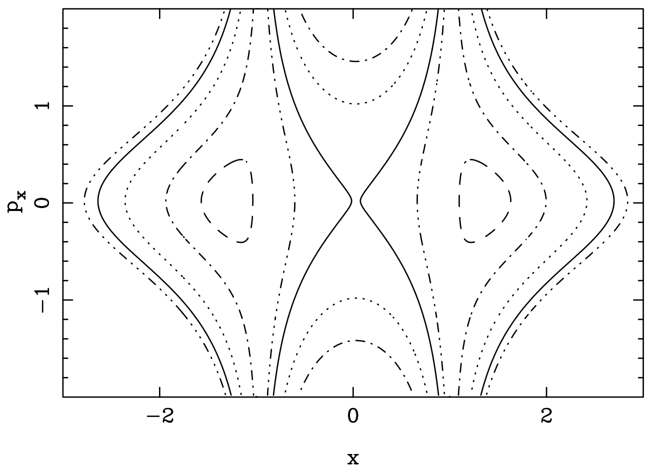

Figure 3. Surface of section plot for several non-periodic trajectories: (a) *P*1 class, dashed curves, (b) *P*2 class, dotted and dashed-dotted curves, (c) *P*3 class, solid and dashed-3-dotted curves.

in action-angle variables. A necessary condition for this is that the available phase space region is compact. However this is not immediately obvious in the case of the two-fixed-centers problem. On a surface of section there are invariant curves, of clearly positive measure, which tend to infinity, in the momentum axis, in an unusual way: invariant curves for consecutive values of the second integral are *not nested*, but they seem to *intersect at infinity* (Figure 3). This behavior can be understood by observing that, due to the form of the Hamilton's equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

we have that

$$p_\xi = \frac{1}{\xi^2 - 1}\frac{\mathrm{d}\xi}{\mathrm{d}t}, \quad p_\eta = \frac{1}{1 - \eta^2}\frac{\mathrm{d}\eta}{\mathrm{d}t} \tag{12}$$

Therefore if the derivatives on the right-hand side are not equal to zero, at least one of the momenta tends to infinity as the moving particle approaches one of the two attracting centers. A simple calculation of surface of section plots shows, indeed, the existence of this kind of trajectories, which we call *asymptotic* collision trajectories. In what follows we show that asymptotic collision trajectories, which are very difficult to be studied numerically due to the ever increasing values of the momenta, form a set of complete measure.

The picture in Figure 3 does not agree with the usual one, where the phase space of an integrable system is foliated into *nested* tori. Drawing a surface of section, however, that does not contain the lines $\xi = 1$ or $\eta = \pm 1$ shows that the invariant curves' segments are joined in such a way, that the foliation picture is recovered (Figure 4). Indeed, we prove in Section 6, through a regularization scheme, that the points at infinity lie on a circle, so that the invariant curves do not really cross. The phase portrait in the regularized coordinates, the distribution of periodic orbits and their relation to the rotation curves will be the topic of the next section.

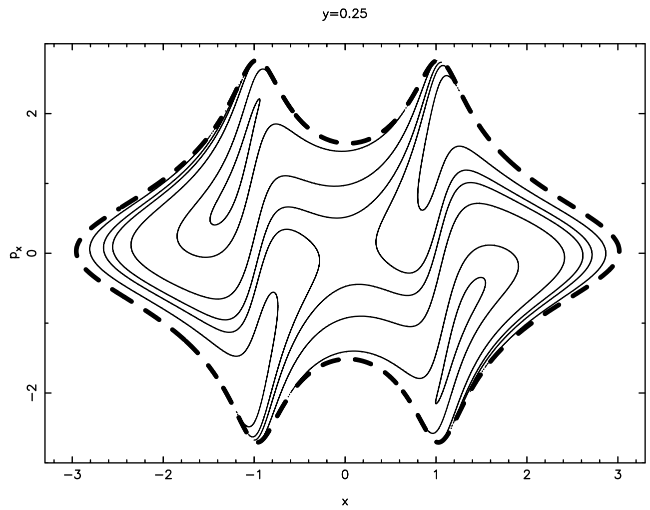

Figure 4. Surface of section at $y = 0.25$, which does not contain the attracting centers. The invariant curves (solid lines) reflect now the foliation of phase space. The dashed curve is the zero-velocity curve.

4. Regularization of Collisions

The equations of motion in $(\xi, \eta, p_\xi, p_\eta)$, as derived from Equation (8), show an artificial invariant manifold at collision

$$\xi = 1, \quad \eta = \pm 1, \tag{13}$$

due to the singularities of the transformation, thus e.g. numerical integration in these variables cannot go beyond such an event. In order to circumvent this problem and using the fact that

$$\xi \geqslant 1, \quad -1 \leqslant \eta \leqslant 1 \tag{14}$$

we apply two 'natural' point transformations to each one of the separated parts of the original Hamiltonian:

$$\xi = \operatorname{ch} u, \quad p_\xi = \frac{p_u}{\operatorname{sh} u}, \quad \eta = \cos v, \quad p_\eta = -\frac{p_v}{\sin v} \tag{15}$$

where by sh() and ch() we represent the functions *hyperbolic sine* and *cosine*. The two integrals of motion in the extended phase space then become

$$\begin{cases} K_u(u, p_u, p_t) = \frac{1}{2} p_u^2 + \alpha \operatorname{ch} u + p_t \operatorname{ch}^2 u \quad \equiv \gamma \\ K_v(v, p_v, p_t) = \frac{1}{2} p_v^2 + \beta \cos v - p_t \cos^2 v \equiv -\gamma \end{cases} \tag{16}$$

The Hamiltonian function is then written as

$$\begin{aligned} \widetilde{K} &= K_u + K_v \\ &= \frac{1}{2} p_u^2 + \alpha \operatorname{ch} u + p_t \operatorname{ch}^2 u + \frac{1}{2} p_v^2 + \beta \cos v - p_t \cos^2 v \equiv 0 \end{aligned} \tag{17}$$

and the equations of motion become:

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}\tau} = \dfrac{\partial \widetilde{K}}{\partial p_t} = \operatorname{ch}^2 u - \cos^2 v \geqslant 0 \\ \dfrac{\mathrm{d}p_t}{\mathrm{d}\tau} = -\dfrac{\partial \widetilde{K}}{\partial t} = 0 \\ \dfrac{\mathrm{d}u}{\mathrm{d}\tau} = \dfrac{\partial \widetilde{K}}{\partial p_u} = p_u \\ \dfrac{\mathrm{d}p_u}{\mathrm{d}\tau} = -\dfrac{\partial \widetilde{K}}{\partial u} = -\alpha \operatorname{sh} u - 2 p_t \operatorname{ch} u \operatorname{sh} u \\ \dfrac{\mathrm{d}v}{\mathrm{d}\tau} = \dfrac{\partial \widetilde{K}}{\partial p_v} = p_v \\ \dfrac{\mathrm{d}p_v}{\mathrm{d}\tau} = -\dfrac{\partial \widetilde{K}}{\partial v} = +\beta \sin v - 2 p_t \cos v \sin v \end{cases} \tag{18}$$

Writing down the transformation from the old to the new canonical coordinates and vice versa, we have

$$u = \operatorname{arch} \xi \tag{19}$$

$$v = \begin{cases} \arccos \eta, & y \geqslant 0 \\ 2\pi - \arccos \eta, & y < 0 \end{cases} \tag{20}$$

$$p_u = \operatorname{sh} u \cos v \, p_x + \operatorname{ch} u \sin v \, p_y \tag{21}$$

$$p_v = -\operatorname{ch} u \sin v \, p_x + \operatorname{sh} u \cos v \, p_y \tag{22}$$

$$x = \operatorname{ch} u \cos v \tag{23}$$

$$y = \operatorname{sh} u \sin v \tag{24}$$

$$p_x = \frac{p_u \cos v \operatorname{sh} u - p_v \operatorname{ch} u \sin v}{\operatorname{ch}^2 u - \cos^2 v} \tag{25}$$

$$p_y = \frac{p_u \sin v \operatorname{ch} u + p_v \operatorname{sh} u \cos v}{\operatorname{ch}^2 u - \cos^2 v} \tag{26}$$

We see that the ambiguity present in Equations (4) and (6) has been removed by using the convention

$$\operatorname{sign} y = \operatorname{sign}(u) \operatorname{sign}(\sin v) \tag{27}$$

and the orbit is *regularly* continued through a collision

$$u = 0, \quad v \in \pi\mathbb{Z} \tag{28}$$

In this way the difficulty in following numerically orbits of the classes $P2$ and $P3$ for long times is relaxed. In Figure 5 we show the same trajectory

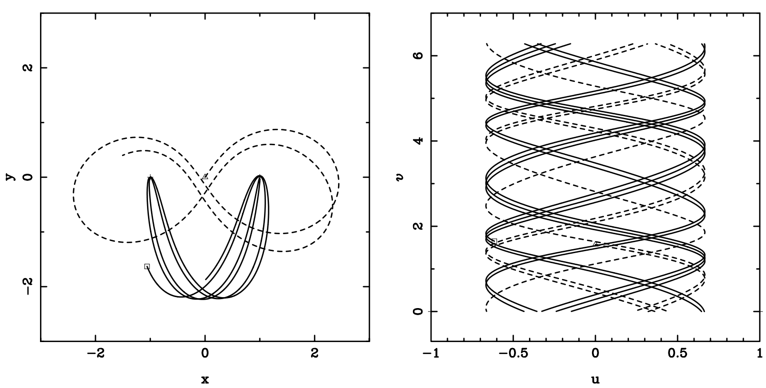

Figure 5. The evolution of a 'typical' quasi-periodic trajectory, 'near' the figure-8 one. The initial position of the first segment ($t_0 = 0.0$, $t_1 = 20$, dotted line) is given by the triangle. The final position of the second segment ($t_2 = 62$, $t_3 = 94$) is given by a square. Notice the qualitative difference between the x–y (left) and u–v (right) plots.

integrated in the 'old' and the 'new' coordinates. It is obvious that the trajectory is evolving quite regularly in the u–v plane, while it undergoes consecutive 'close-encounters' in the x–y plane.

5. Periodic and Quasi-periodic Motion

The equations of motion of the two-fixed-centers problem can be solved by quadratures, which involve elliptic integrals and functions (Equations (2.17) and (2.18) of Strand and Reinhardt, 1979). From the solutions we can calculate the periods of motion along the two axes, η and ξ, and from them their ratio, i.e. the 'rotation number', in the following way.

5.1. MOTION ALONG THE η-AXIS

The form of the solution of the equation of motion along the η-axis depends on γ and E. For the values of the constants of motion that correspond to the class $P2$, the solution is given by Equation (2.17b) of Strand and Reinhardt (1979), which is a function of the elliptic function cn,

$$\eta(\tau) = \pm\eta_1 \,\mathrm{cn}\left[(-2E)^{1/2}(\eta_1^2 + \eta_2^2)^{1/2}(\tau - \tau_0), k\right] \tag{29}$$

where $\eta_1 = 1$ and $\eta_2 = \sqrt{\gamma/E}$ and the argument of the function, k, is given by the relation

$$k = \left(\frac{\eta_1^2}{\eta_1^2 + \eta_2^2}\right)^{1/2} \tag{30}$$

Therefore the period, T_η, of the motion along the η-axis is

$$T_\eta(\gamma, E) = \frac{4}{\sqrt{-2E(1 + \gamma/E)}} K(k) = \frac{4}{\sqrt{-2E - 2\gamma}} K\left(\frac{1}{\sqrt{1 + \frac{\gamma}{E}}}\right) \tag{31}$$

5.2. MOTION ALONG THE ξ-AXIS

In the same way we see that the solution of the equation of motion along the ξ-axis for the class $P2$, Equation (2.18a), for values of γ and E corresponding to trajectories of class $P2$, is a function of the elliptic function sn *squared*, with the argument of the function, k, given by

$$k^2 = \frac{E - \gamma + 2\sqrt{1 - E\gamma}}{4\sqrt{1 - E\gamma}} \tag{32}$$

After some calculations we find that the period, T_ξ, of the motion along the ξ-axis is given by the relation

$$T_\xi(\gamma, E) = \frac{2}{\sqrt{2}\sqrt[4]{[1-E\gamma]}} \mathrm{K}\left(\frac{\sqrt{E-\gamma+2\sqrt{1-E\gamma}}}{2\sqrt[4]{[1-E\gamma]}}\right) \tag{33}$$

It is quite evident that, within a certain range of values of γ and E (see Figure 6) *any* ratio (or 'rotation number')

$$R(\gamma, E) = T_\xi(\gamma, E)/T_\eta(\gamma, E) \tag{34}$$

between the two periods can be achieved. Therefore the trajectories belonging to the class $P2$ are either periodic or quasi-periodic. In particular, since the rational numbers are dense in the set of real numbers but of zero measure, we can conclude that we have two important sets of trajectories.

(1) A dense set of measure zero of initial conditions with $R(\gamma, E) \in \mathbb{Q}$ yielding periodic motion (Lissajous-figures in the (u, v), (u, p_v), (p_u, v), (p_u, p_v) planes respectively). A special case with $R = 2$ are the figure-8 orbits.

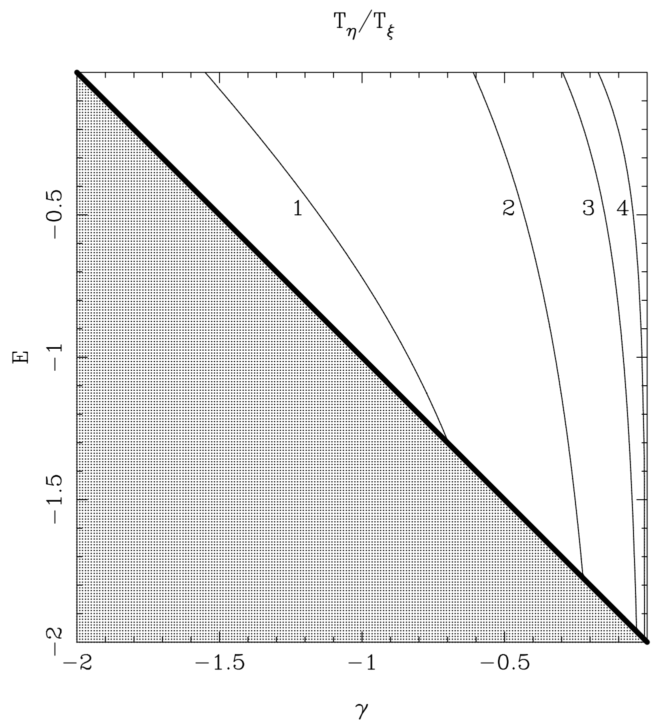

Figure 6. The ratio of periods along the two degrees of freedom as a function of the two integrals of motion, γ and E.

(2) A dense set of positive measure of initial conditions with $R(\gamma, E) \in \mathbb{R} \backslash \mathbb{Q}$ involving quasi-periodic motion. Note that each of these orbits may enter any arbitrarily small neighborhood of both of the foci after sufficiently long time. According to the terminology we introduced in Section 3, these are the 'asymptotic collision orbits.'

A similar reasoning holds for the region $P3$, except that one single orbit may collide or nearly collide only with*one* of the centers. The statements for the existence of periodic and quasi-periodic orbits, apart from collisions, hold also for the region $P1$.

6. Density and Measure of Pure Collision Orbits

In order for an orbit to be a pure collision one, it must contain either m_1 or m_2. The first case in the three coordinate systems used here is equivalent to

$$(x, y) = (-1, 0), \quad (\xi, \eta) = (1, -1) \quad \text{or} \quad (u, v) = (0, \pi + 2k\pi),\ k \in \mathbb{Z} \tag{35}$$

while the second is equivalent to

$$(x, y) = (1, 0), \quad (\xi, \eta) = (1, 1) \quad \text{or} \quad (u, v) = (0, 2k\pi),\ k \in \mathbb{Z} \tag{36}$$

For symmetry reasons we only treat collision with m_2. At the very moment of collision we have (from Equations (22), (23), (26) and (27))

$$p_u^2 + p_v^2 = 4 \tag{37}$$

and

$$p_u = \pm\sqrt{2(E + \gamma + 2)} \tag{38}$$

$$p_v = \pm\sqrt{2(-E - \gamma)} \tag{39}$$

from the integrals K_u and K_v respectively (remember that $p_t = -E$!). Therefore, in momentum space, the initial conditions of true collision trajectories lie on a circle.

Restricting v to $[0, 2\pi]$ one thus has determined the four possible points of collision in phase space by a single pair (γ, E). Now pick out one of them, since the others can be treated similarly through symmetry, and consider a finite length segment of the pseudo-time (τ) evolved orbit, passing through it. Then one can show that, by varying γ and E within the region of $P2$, the resulting bundle of curve segments spans a 3-D volume of positive 3-D measure in 4-D space. Now, since any orbit can be regarded as the countable union of finite length curve segments, the set of initial conditions leading to collision can be represented as a countable union of 3-D volumes. This union is of positive 3-D measure in any 3-D subspace projection but of zero 4-D measure in 4-D space.

Moreover, when dropping the set of measure zero, for which $R \in \mathbb{Q}$, from 4-D phase space, one finds that any orbit fills densely a smooth surface, transverse to all the above mentioned 3-D sets, exhausting the allowed ranges of motion in each phase space variable respectively. Therefore the 3-D sets of positive 3-D measure lie dense within a compact subset of 4-D space around the point of collision, provided a neighborhood of the point $P012$ (which corresponds to exact collision) is avoided! Summarizing one thus has a dense set of measure zero of initial conditions in 4-D phase space of collision orbits.

It should be noted, however, that if the two attracting centers are bodies with physical, non-zero dimensions, and not mathematical points, then the asymptotic collision become true collision orbits as well, so that they have a complete measure in full 4-D space.

7. Summary – Discussion

In this paper we have presented some 'annoying' features of the planar two-fixed-centers problem and we have proposed ways to circumvent them.

We have shown that (bounded) trajectories restricted in configuration space region that contain either attracting center (classes $P2$ and $P3$) cannot be written in action-angle variables, although the dynamical system is separable, because all trajectories in these regions are either collision (of measure 0) or asymptotic collision (of complete measure) orbits. As a result, all trajectories approach arbitrarily close at least one of the two attracting centers, so that the allowable region of phase space is not compact. The above problem does not appear for trajectories of class $P1$. Since almost all of the applications of the 2FC problem, up to now, were on the motion of artificial satellites, whose trajectories belong to the class $P1$ (since the two centers are in this case located both *inside* the Earth), the above 'irregular' behavior was so far not appreciated.

By selecting as surface-of-section a phase space plane not containing the attracting centers, we have found that invariant curves join in a smooth way, so that an appropriate canonical transformation should 'regularize' collisions and asymptotic collisions. This transformation is given in Section 4. In the new variables the trajectories are continued through collisions in a consistent way, so that the numerical integration of quasi-periodic trajectories (which are the asymptotically collisions orbits) can be computed without any problem.

We have calculated in closed form the rotation number as a function of the constants of the motion, so that we can determine the initial conditions for periodic trajectories of any resonance.

Finally we have shown that true collision orbits of class $P2$ and $P3$ are of complete measure in a 3-D sub-manifold but of zero measure in the full 4-D space. However if the attracting centers are physical bodies and not mathematical points, then all trajectories become collision orbits and in a finite time are terminated on either attracting center.

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