The "Third" Integral in the Restricted Three-Body Problem Revisited

Harry Varvoglis, Kleomenis Tsiganis, and John D. Hadjidemetriou

Department of Physics, University of Thessaloniki, GR-541 24 Thessaloniki, Greece

Abstract. In 1964 M. Hénon and, independently, V. Szebehely with G. Bozis presented the first numerical results, indicating the existence of a "new" local integral of motion in the circular restricted three-body problem. The first terms of an asymptotic expansion of this integral were later calculated by Contopoulos [1]. Several years later, the Celestial Mechanics astronomical community started to develop a very successful theory on local integrals of motion in the restricted three-body problem, which however in the jargon of this field are called *proper elements* and are related to known analytical approximate solutions. The calculation of proper elements is based on the implicit assumption that the orbit traced by a planet (major or minor) is nearly-regular. Here we show that this method is also applicable, albeit partly, in a special case of chaotic motion in the Solar System, known as "stable chaos". Thus, the existence of an additional local integral of motion in the elliptic restricted three-body problem is responsible for the phenomenon of stable chaos.

1 Introduction

In 1964 the Laboratory of Astronomy of the University of Thessaloniki hosted IAU Symposium 25. This meeting was devoted to the interaction between astronomers working on two widely different fields of Dynamical Astronomy, namely Galactic Dynamics and Celestial Mechanics, in the hope that the methods used traditionally in one of the fields could prove useful in the other. Indeed, several papers presented in this meeting followed the above line. In two of them Hénon [2], on the one hand and, independently, Szebehely and Bozis [3] on the other, reported that they had found indications for the existence of a further integral of motion in the planar circular restricted three-body problem (a two-degrees of freedom dynamical system), besides the well known Jacobi integral.

Subsequently Contopoulos [1] showed how this integral could be constructed in a series form through an algorithm similar to the one he had proposed already [4] for the "third" integral in the case of a galactic type potential, in which (series) the zeroth order term is the angular momentum. At the same time Bozis [5] [6] studied extensively the properties of this new integral, as well as the computation, through its use, of "generalized" elements of motion (e.g. eccentricity, see next paragraphs).

Since Poincaré had shown that the three-body problem is non-integrable, it is obvious that this integral can only be a "local" (non-isolating) one. Therefore one should inquire in which regions of phase space this integral may be applied, as it was initiated by Bozis [6]. These regions should be called "regular", since the corresponding dynamical system has two degrees of freedom and, therefore, in the regions where there exist two integrals of motion, it behaves like an integrable one.

A three-dimensional elliptic orbit of the two-body problem is uniquely defined by three quantities, the three *elements* of the orbit a, e and I, where I is the *inclination* of the plane of the orbit with respect to a "reference" plane, a is the semi-major axis of the ellipse and e the *eccentricity.* In what follows we consider the motion of massless test-particles (i.e. asteroids) relative to a massive central body (i.e. the Sun) of mass M. The orbital elements of the minor planet are related to the energy, E and the angular momentum, h, of its orbit, through the relations

$$a = -\frac{G\,M}{2E} \tag{1}$$

$$e = \sqrt{1 + \frac{2Eh^2}{G^2\,M^2}} \tag{2}$$

For elliptic motion, the orbital energy, E, has to be negative.

It is worth to note that the two-body problem is an intrinsically degenerate dynamical system [9], a property that becomes obvious if we write the corresponding Hamiltonian in action-angle variables. One possible set of action angle variables in this case are the well known *modified Delaunay variables*, defined through the relations

$$\Lambda = \sqrt{G\,M\,a} \qquad \lambda = \varpi + l \tag{3}$$

$$\Gamma = \Lambda(1 - \sqrt{1-e^2}) \quad \gamma = -\varpi \tag{4}$$

$$Z = \Gamma(1 - \cos i) \qquad \zeta = -\Omega \tag{5}$$

where the angles Ω, ϖ and l are the three Euler angles: the first two define the orientation of the ellipse in space and the third one the position of the the asteroid on the ellipse. In Celestial Mechanics the various angles have their own names: Ω is the *longitude of the ascending node* of the orbit, $\varpi = \Omega + \omega$ is the *longitude of the pericenter* and $\lambda = \varpi + l$ is the *mean longitude.* The *mean anomaly*, l, is related to time through the relation $l = n\,t$, where n is the *mean motion of the planet*, i.e. its mean angular frequency around the massive central body. The Hamiltonian of the two-body problem, written in the above variables, becomes simply

$$H = -\frac{G^2\,M^2}{2\,\Lambda^2} \tag{6}$$

i.e. it depends only on the action corresponding to the energy, which, according to eq. (1), depends only on the semi-major axis.

The two-body problem is only a simple approximation of a planet's motion around the Sun. A better approximation is the restricted three-body problem. In this model a massless particle is moving in the gravitational field of two bodies, a central massive primary of mass M (the Sun) and a perturbing planet of mass m (say Jupiter). Moreover, the motion of the perturbing planet around the Sun is a Keplerian closed orbit (i.e. either a circle or an ellipse). The trajectory of the massless body is not anymore an ellipse, due to the perturbations induced by the planet. However, due to the small mass of the perturber relative to the Sun and for relatively large separation between the asteroid and the perturber, the trajectory can be described by means of the *osculating elements*, i.e. instantaneous values of the variables $a(t)$, $e(t)$ and $I(t)$, defined as the elements of an ellipse that is tangent to the real orbit at time t. The process is very easily implemented, since it reduces to the calculation of the elements of the orbit from the instantaneous values of the energy and the angular momentum (which, of course, are not anymore constants in the case of the three-body problem).

2 The way things might have happened

2.1 Ordered trajectories

From the form of the Hamiltonian alone and some educated guesses, one could relatively easily arrive at the form of the third integral, for the existence of which Hénon, Szebehely and Bozis had found numerical evidence, as follows. In the restricted three-body problem the Hamiltonian can be "split" into two parts, one of order zero with respect to the *mass ratio*, $\mu = \frac{m}{M+m}$, and one of order unity. In modified Delaunay variables the zeroth-order term depends only on Λ, while the other two actions appear only in the first order term, which therefore may be considered as a "perturbation". Thus, we have again a case of degeneracy, similar to the one appearing in the two-body problem. Due to this degeneracy, the Fourier expansion of the perturbation contains terms that do not depend on the angle λ. Therefore, if one ignores the terms involving λ and λ' [1], which become important only when they are almost resonant, the osculating semi-major axis, a, is constant, a famous result known as the *Laplace-Lagrange linear theory of secular motion.* Then E is constant to a linear approximation as well, since it depends only on the osculating semi-major axis through eq. (1). As a consequence and, in view of eq. (2), the osculating eccentricity, e, is, to a linear approximation, a function of h only, i.e. e depends, essentially, only on the angular momentum. Therefore it is natural to expect that, if one would attempt to calculate a "third" integral *for the full, non-linearized problem* as a series, using as a small parameter the mass ratio, μ, the zero-order term should be the angular momentum of the massless body on its (unperturbed) orbit around the central body. This is exactly the method used by Contopoulos [1]. In the same linear approximation as for a, the osculating eccentricity of the asteroid is given by

$$e^2 = e_f^2 + e_P^2 + 2\, e_f\, e_P\, \cos(g_P\, t + \beta_P), \tag{7}$$

[1] Note that by a prime we denote the angles of the perturbing planet

where e_f, e_P, g_P and β_P (the phase at $t = 0$) are constants. In particular e_f (usually called *forced eccentricity*) and g_P (*proper frequency*) depend only on a and μ, while e_P is the constant amplitude of variation of the osculating eccentricity. In the full, non-linearized problem, e_P can be calculated through an algorithm similar to the one used by Contopoulos [1], and is called the *proper eccentricity.*

Since the circular restricted three-body problem is a two-degrees of freedom autonomous dynamical system, the existence of a second integral of motion would imply integrability. In this case all trajectories would be ordered and the secular solution would always remain $\mathcal{O}(\mu)$ close to the real solution. Note that eq. (7) is the simplest secular theory of Celestial Mechanics (e.g. see Yuasa [7] or Milani and Knežević [8]). This result can be generalized for forms of the restricted three-body problem with more than two degrees of freedom, such as the elliptic (where the orbit of the perturber is an ellipse) or the three-dimensional (where the massless body moves outside the plane of the orbit of the perturber). In these cases one would need to calculate further integrals of motion, in the same spirit. As far as the total number of integrals is equal to the number of degrees of freedom of the corresponding (autonomous) dynamical system, all trajectories would be ordered. In this way we see that the three proper elements of the trajectory (or the associated modified Delaunay variables) constitute a set of action variables (and hence integrals of motion) of the *secular three-body problem.*

2.2 Chaotic trajectories

The proper elements of ordered trajectories of asteroids are calculated through the secular theory at any desirable level of accuracy. However we know, from the work of Poincaré, that the restricted three-body problem does not admit any further integrals of motion, *analytic* in any variables. Therefore the corresponding dynamical system is non-integrable and the integrals in series form calculated through the method of Contopoulos (or some secular theory) can only be non-isolating, local ones. Hence in the vicinity of orbital resonances between the test-particle and the perturber (i.e. resonances between the angles λ and λ') the secular theory should fail, as a result of the small divisors problem and the appearance of chaotic motion. This means that all specific models of the restricted three-body problem (e.g. circular, elliptic or three-dimensional) should possess chaotic phase-space regions, besides the ordered ones. What can we say on the properties of chaotic trajectories? This problem was attacked by many authors through extensive numerical calculations, according to the available, at any period, computing power. The first model studied was the simplest one, namely the planar circular restricted three-body problem.

Soon it was realized, however, that this model does not represent the generic case, since it corresponds to an autonomous dynamical system with two degrees of freedom. But in this class of dynamical systems Arnold's diffusion (see e.g. [9]), which might play an important role in solar system dynamics, cannot be taken into account. Therefore, if we would like to consider a "generic" model for three-body dynamics, we should have at least three degrees of freedom! Consequently one should use as a "generic model" either the elliptic planar restricted or the circular three-dimensionl restricted problem and not the planar circular. This was done by Contopoulos, who calculated the form of the "third" integral in the case of the three-dimensional restricted three-body problem [10] and the planar elliptical three-body problem [11].

The difference between the circular restricted three-body problem, on one hand, and the three-dimensional or elliptic restricted problem, on the other, is qualitative[2]. In both cases there exists a global (isolating) integral, which is the Jacobi integral in the first and the Hamiltonian of the extended phase-space in the second. But in the first case the situation is clear-cut: a specific trajectory is either ordered (if an additional local integral exists) or chaotic (if no local integrals exist). In the second case, however, there may exist from none to two local integrals of motion [12]. Two local integrals imply regular behavior and ordered trajectories, for which the secular solution would be an accurate approximation. The other two sub-cases correspond to chaotic motion, but with significant differences. If no local integrals exist, the chaotic trajectory covers densely a sub-manifold of the phase-space, defined by the constant "energy" surface. If one local integral exists, then the trajectory lies on a manifold which is the cartesian product of a two-dimensional torus with an annulus [18] (see Fig. 1). The motion on the two-torus corresponds to the ordered part of the trajectory, originating from the existence of the two integrals, while the motion on the annulus corresponds to the chaotic part.

In the case where no local integrals exist, the motion is "fully" chaotic, i.e. macroscopically it is equivalent to a random walk. Therefore, one might use methods of statistical mechanics (e.g. a Fokker-Planck-type equation) in order to describe the evolution of a set of initial conditions as a diffusion process in the elements space. Since, according to what has been already said, the semi-major axis is constant to a linear approximation, we can select as a dependent variable either the eccentricity or the inclination. The eccentricity is our first choice, since it is intimately related to the escape of asteroids from the main belt.

It is easy to see that e increases on the average, since if we consider the chaotic motion as a random walk in eccentricity space, there is a reflecting wall at $e = 0$! Moreover, as e increases the resonances begin to overlap and chaotic motion becomes dominant. Therefore asteroids in fully chaotic trajectories follow more and more elongated orbits, until they hit a planet and are removed from the distribution. An analytic theory for the diffusion of asteroids was developed by Murray and Holman [13] and was recently applied, with considerable success, for the estimation of the age of the Veritas family of asteroids [14].

In the case where one local integral exists, the motion is "partially" chaotic, which means that some degrees of freedom are evidently chaotic and some appear as being ordered. From extensive numerical experiments it is relatively

[2] The 2-D elliptic and the 3-D circular problem are also by no means equivalent to each other.

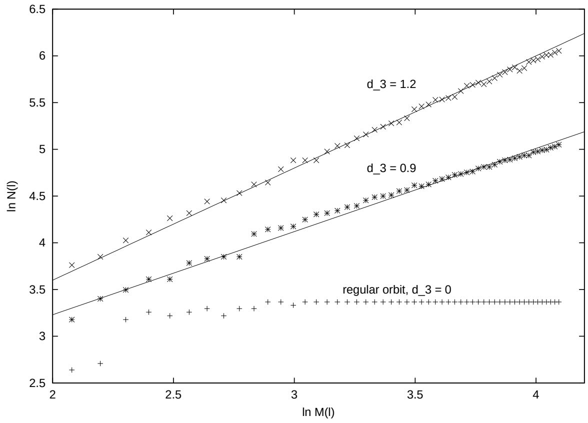

Fig. 1. Calculation of the number of integrals of three trajectories, one ordered and two stable chaotic, in the region of the 12:7 orbital resonance (from [15]). According to the theory, if we partition a 3-D space in M^3 bins of side l, N of which are occupied by a trajectory, then we have that $\log N(l) \sim d\_3 \log M(l)$, where $d\_3 = 3 - d$, and d is the number of integrals. The regular orbit yields $d\_3 = 0$, i.e. $d = 3$, while stable-chaotic orbits have $d\_3 \approx 1$, i.e. $d \approx 2$

straightforward to show that the evolution of a is chaotic, while e and I change almost quasi-periodically with time, their proper values being almost constant [16] [17] [18] (Fig. 2). But, according to the secular theory, a only undergoes bounded erratic oscillations and does not change secularly, unless of course the trajectory escapes from the (non-isolated) region of the elements' space, where it is restricted by the level surfaces of the local integral. Since the usual way for the classification of trajectories is through the calculation of the Maximal Lyapunov Number, which in this case is positive, "partially chaotic" trajectories could be named, as well, "stable chaotic". Since for a stable chaotic trajectory e_P does not increase on the average, there are no collisions with other planets and, therefore, no escapes.

Extensive numerical work has shown that another important property of a phase-space region, besides the existence of local integrals of motion, is the existence or not of simple-periodic resonant trajectories. Although in the restricted circular three-body problem all orbital resonances with Jupiter correspond to periodic trajectories, this is not true for the elliptic problem. In general, orbital resonances do not correspond to periodic trajectories, unless their period is an exact multiple of Jupiter's revolution period [16]. Thus, the chaotic regions of phase space (i.e. the resonances' zones), in the planar elliptic (or the three-dimensional circular) restricted three-body problem, can be classified into three classes as follows, according to the type of trajectories they contain and the existence or not of periodic trajectories [16] [17] [18].

Stable chaotic regions constitute the first class. In such a region the evolution of trajectories is not diffusive. Chaotic trajectories are semi-confined by the level

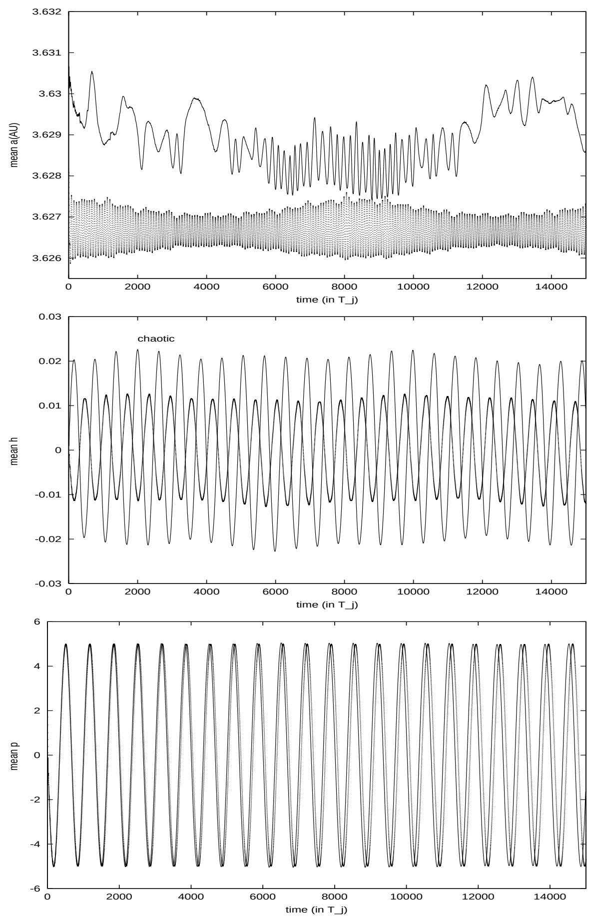

Fig. 2. The elements a (top), $h = e \sin \varpi$ (middle) and $p = I \sin \Omega$ (bottom) are given, as functions of time, for one regular and one stable-chaotic orbit of the elliptic three-body problem in the vicinity of the 12:7 orbital resonance (from [15]). The unit of time is the revolution period of Jupiter, $T_J \approx 11.86$ yr. It is easy to realize the different character of the motion between these two orbits, by monitoring the behavior of a. On the other hand, one cannot decide whether an orbit is regular or chaotic by just observing the graphs of h or p

surfaces of the local integrals. Since, however, these surfaces are non-isolating, the trajectory eventually escapes from such a region through the "holes" of the "invariant" manifold. After such an escape, the eccentricity increases steeply. Numerical experiments have shown that the typical time-scale, T, for escape through this process is $T \sim 1$ Gyr and can even exceed the age of the solar system (5 Gyrs), depending on the specific resonance.

Fully chaotic regions are divided into two classes, according to whether they support simple periodic orbits or not. If there are no periodic orbits, the evolution is diffusive, i.e. a trajectory undergoes many small "jumps" in eccentricity. This case is the one that can be described successfully through a diffusion equation and its typical time-scale, as can be calculated by the values of the diffusion coefficient, is of the order of 100 Myrs $< T <$ 1,000 Myrs (again, depending on the specific orbital resonance).

If there exist periodic orbits, then the evolution of chaotic trajectories is "fast" and intermittent, as the trajectory from time to time follows the unstable periodic orbit. This is the kind of motion found by Wisdom [19] and Hadjidemetriou [20]. The typical time-scale for the "jumps" is of the order of $5 \cdot 10^5$ yrs, while the escape time is of order $10^5 < T < 10^6$ yrs. There are only 5 such resonances in the phase-space region that corresponds to the main asteroid belt, in both the elliptic and the three-dimensional restricted three-body problems. These are the 2:1, 3:1, 4:1, 5:2 and 7:3 orbital resonances with Jupiter. Since the more well-known Kirkwood gaps lie exactly at these resonances, one arrives easily at the conclusion that the existence of a periodic trajectory is the common factor that differentiates between orbital resonances, associated with a Kirkwood gap, and those that are not.

Summarizing, we can say that stable chaos is the observational manifestation of the existence of a local integral of motion, while the Kirkwood gaps appear at resonances where periodic orbits exist, in the elliptic or the three-dimensional restricted three-body problem.

3 The way things really happened

Unfortunately, the evolution of ideas in science does not always follow the "obvious" path. The applicability of local integrals of motion presents another case of misunderstanding between theorists and applied-oriented astronomers. The scientific community of Celestial Mechanics did not capitalize on the work of Bozis and Contopoulos, related to the existence of local integrals of motion and the calculation of "primitive proper elements". Instead, for quite some time, the calculation of proper elements was only used for objects that move far away from the main resonances, where secular theory could apply.

Things started to change in the 1980's, when algorithms for the calculation of the maximal LCN were made available and Wisdom [19] found the "intermittent" behavior of the osculating eccentricity in the vicinity of the 3:1 resonance, which is characteristic of the existence of an unstable periodic trajectory. However, since as a rule only the maximal LCN was calculated, there was no way to differentiate between regions where none or one local integral exists. That is why the chaotic motion in the regions where local integrals exists was considered "peculiar" and termed *stable chaos*.

The first to point out that stable chaotic motion is not "fully chaotic" were Varvoglis and Anastasiadis [21]. This idea was subsequently elaborated in a series of papers by Tsiganis, Varvoglis and Hadjidemetriou [16] [17] [18]. In these papers it is shown, through the computation of autocorrelation functions, that stable-chaotic trajectories have almost constant proper elements, i.e. they possess local integrals of motion (see Fig. 1), and lie at the border between fully chaotic and regular phase-space regions. Consequently, stable-chaotic orbits represent cases of *sticky motion* in G and H (i.e. essentially eccentricity and inclination) and chaotic motion in L (i.e. semi-major axis), a type of motion for which no analogue exists in two-dimensional dynamical systems. The subsequent numerical calculation of the number of integrals, preserved by a large number of trajectories of the elliptic restricted three-body problem [15], confirmed this picture. In this way today we arrived finally, after thirty-six years, in the “re-discovery” of the work of Contopoulos-Bozis and its connection to proper elements, by understanding the phenomenon of stable chaos and its relation to local integrals of motion.

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