Logic Homework 1

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Abstract

We present a proof of the unique readability of prepositional sentences that relies primarily on an induction argument on the length of such sentences.

Definition 1 (Words) *Let* $A = \{0, 1, \neg, \wedge\}$. *Then* A *combined with concatenation form a monoid* A^*. *Let the empty word be* ϵ. *Let* $|a|$ *denote the length of a word* $a \in A^*$ *with* $|\epsilon| = 0$.

Definition 2 (Sentences) *Let*

$$\mathrm{Sent}_0 = \{p_i | p_i \in \{0, 1\}, i \in \mathbb{N}\}$$

be the set of atomic variables.

Now inductively define Sent_n *as*

$$\mathrm{Sent}_{n+1} = \{\neg\varphi | \varphi \in \mathrm{Sent}_n\} \cup \{\wedge\varphi\psi | \varphi, \psi \in \mathrm{Sent}_n\}$$

and finally

$$\mathrm{Sent} = \bigcup_{n \in \mathbb{N}} \mathrm{Sent}_n\,.$$

Note that $\forall\varphi \in \mathrm{Sent}, \varphi \in A^*$.

Lemma 1 *The smallest set S of expressions with*

1. $\mathrm{Sent}_0 \subset S$
2. *If $\varphi \in S$ and $\psi \in S$, then $\neg\varphi \in S$, $\neg\psi \in S$ and $\wedge\varphi\psi \in S$.*

is precisely the set Sent.

Lemma 2 *If P is a set of sentences and*

1. $\mathrm{Sent}_0 \subset P$
2. *P is closed under $\neg$ and $\wedge$.*

then $\mathrm{Sent} \subset P$.

Lemma 3 *Each sentence is in one of exactly these forms:*

1. *p_i where $p_i \in \mathrm{Sent}_0$*
2. *$\neg\varphi$ where $\varphi \in \mathrm{Sent}_0$*
3. *$\wedge\varphi\psi$ where $\varphi, \psi \in \mathrm{Sent}_0$*

Moreover, φ and ψ are unique.

Proof 1 *First we approach the distinct forms of sentences in* Sent. *Notice that these cases are mutually exclusive, for as words in A^* their first character will differ. Now let P be the set of sentences such that one of these properties hold. Note that* $\mathrm{Sent}_0 \in P$ *by (1). Now let $\varphi \in P$. Then by (2) $\neg\varphi \in P$. Similarly for $\varphi, \psi \in P$ we have by (3) that $\wedge\varphi\psi \in P$. Therefore the conditions of Lemma 2 are satisfied implying that* $\mathrm{Sent} \subset P$.

Claim 1 *Let* $\varphi_0, \varphi_1, \ldots, \varphi_n$ *and* $\psi_0, \psi_1, \ldots \psi_m$ *be in* Sent *with each* φ_i *and* $\psi \in$ Sent. *If* $\theta = \varphi_0\varphi_i \cdots \varphi_n = \psi_0\psi_1 \cdots \psi_m$ *then* $n = m$ *and* $\varphi_i = \psi_i$.

The proof of the claim proceeds by induction on the length of θ. *If* θ *is of length* 1 *then the result holds as this may only be an atom* $p_i \in \mathrm{Sent}_0$.

Now assume this property holds for all sentences of a given length l *and let* $|\theta| = l+1$. *Note that both* φ_0 *and* ψ_0 *are in* Sent *and therefore from earlier we know it must be one of the forms listed above. Thus we may find words* $u_a, u_b, v_a, v_b \in A^*$ *such that* $\varphi_0 = u_a u_b$ *and* $\psi_0 = v_a v_b$ *with* $|u_a| = |v_a| = 1$.

Therefore by cancellation $\varphi_b\varphi_1 \cdots \varphi_n = \psi_b\psi_1 \cdots, \psi_m$ *is of length* $|l|$. *Further, since* θ *is in one of the forms above,* $u_a = v_a$ *also by cancellation and it must be either an atom,* $\wedge$ *or* $\neg$. *The only way it may be an atom is if* $|\theta| = 1$ *which is the base case.*

In each other case we have that $u_b\varphi_1 \cdots \varphi_n = v_b\phi_1 \cdots \phi_n$ *is a sentence, that is a member of* Sent *as above, of length* $|l|$. *Therefore the induction hypothesis applies and we must have* $n = m$, $u_b = v_b$, $\varphi_i = \phi_i$ *for* $1 \leq i \leq n$. *This proves the claim and therefore the lemma.* $\boxed{\text{Q.E.D.}}$