Logic Homework 1

D. Dakota Blair

September 8, 2009

Abstract

A presentation of solutions to the second homework.

Problem 2.1

How big is the object class of Propositional Logic?

The object class of Propositional Logic is the space of all truth assignments for the set Sent_0. Since this set is countable, and there are two valid truth assignments for each prepositional variables, each truth assignment can be thought of as a mapping from $\mathbb{N}$ to $\{0, 1\} = 2$. The set of such mappings is precisely the object class of Prepositional Logic and is therefore uncountable by diagonalization.

Problem 2.2

How many sentences are there? That is, what is the size of Sent**?**

Elements of Sent (sentences) are words from the alphabet $A = \{0, 1, \neg, \wedge\}$ of finite length, a consequence of the inductive nature of the construction of Sent. The set of all words of finite length in this alphabet A^* contains Sent. Therefore Sent is countable because A^* is countable. To see the countability of A^* rename $\neg$

and $\wedge$ as $2, 3$ respectively. Then each element of A^* is a dyadic number, a subset of the rationals, when preceeded by a decimal.

Problem 2.3

Express $\vee$ and $\rightarrow$ in terms of $\neg$ and $\wedge$. Also present a truth table for XOR **and express it in terms of $\neg$ and $\wedge$.**

Let $p, q \in \mathrm{Sent}_0$.

$$p \vee q = \neg(\neg p \wedge \neg q)$$

$$p \rightarrow q = \neg(p \wedge \neg q)$$

$$p \,\mathrm{XOR}\, q = \neg(\neg p \wedge \neg q) \wedge \neg(p \wedge q)$$

$p \,\mathrm{XOR}\, q$	p	q
0	0	0
1	0	1
1	1	0
0	1	1

Problem 2.4

How many n-ary connectives are there?

There are as many as there are truth assignments of n prepositional variables. This is 2^n which may be seen as a consequence of the discussion of problem 2.1.

Problem 2.5

Provide a reasonable definition of Conjunctive Normal Form (CNF) Prove that all sentences of Prepositional Logic may be expressed in CNF.

Definition 1 (Conjunctive Normal Form) *An iterated conjunction of disjunctions of literals.*

Theorem 1 *Every Propositional Logic sentence has a CNF.*

Proof 1 *Given a Prepositional Logic sentence φ, its (finite) truth table T may be generated consisting of columns labeled φ and p_i for as many prepositional variables appearing in φ. If each row represents a distinct truth assignment for φ, then we may form the CNF of φ by the following method:*

1. *For each true row form a disjunction of (p_i) and $(\neg p_i)$ for each relevant $0 \leq i \leq k$, choosing $(\neg p_i)$ when the entry for p_i is 0 and (p_i) otherwise.*

2. *Do the same for each false row, except choose (p_i) when the entry for p_i is 0 and $(\neg p_i)$ otherwise.*

3. *Let ψ be the the iterated conjunction of these clauses.*

This construction of ψ is logically equivalent to φ. Assume φ differs from ψ for a given truth assignment $\tau : \{1, 2, ...k\} \rightarrow \{1, 0\}$ of $\{p_i\}_{i=1}^{k}$. If ψ is true given τ, then there is no disjunction ψ_0 for which τ fails. This contradicts our construction as a failing disjunction for each false row is included in the CNF of φ. If ψ is false given τ, then there is a disjunction ψ_0 for which τ fails. Let the truth assignment which satisfies ψ_0 be $\rho : \{1, 2, ...k\} \rightarrow \{1, 0\}$ of $\{p_i\}_{i=1}^{k}$. Then ψ_0 will be false for τ only if $\tau = \rho$ which contradicts our construction. Otherwise there is a j for which ρ and τ differ, so τ will satisfy at least the literal x_j in ψ_0 and therefore τ satisfies ψ_0, contradicting the assumption. $\boxed{\text{Q.E.D.}}$

Problem 2.6

Find a basis with a single binary connective.

Definition 2 (NAND) *Let* $p, q \in \mathrm{Sent}_0$*. Define a binary connective* NAND *as*

$$p \,\mathrm{NAND}\, q = \neg(p \wedge q).$$

It can be show this single connective forms a basis for Propositional Logic by emulating the basis $\{\neg, \wedge\}$.

$$\begin{aligned} \neg p &= p \,\mathrm{NAND}\, p \\ &= \neg(p \wedge p) = \neg p \end{aligned}$$

$$\begin{aligned} p \wedge q &= (p \,\mathrm{NAND}\, q) \,\mathrm{NAND} (p \,\mathrm{NAND}\, q) \\ &= \neg(p \,\mathrm{NAND}\, q) = \neg\neg(p \wedge q) = p \wedge q \end{aligned}$$