# Logic Homework 4

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**Abstract**

A presentation of solutions to the fourth homework.

## Problem 4.1

**Theorem** (Compactness of Propositional Logic)**.** *A set of sentences has a model if and only if every finite subset has a model.*

**Corollary** (Consequence is finitary)**.** *A statement that follows from a set of sentences in prepositional logic follows from a finite subset of them.*

**Corollary.** *The four-color problem is finitary. That is, only finite maps need be considered.*

*Proof.* Note that a given (possibly infinite) colored map $a$ may be checked for the four-color property by a (possibly infinite) set $M_a$ of sentences. Define $C$, the set of colors used in a colored map $a$ as

$$C = \{x | x \text{ is a color used in colored map } a\}.$$

Then for a state $S_\alpha \in a$ define a map $\text{Color} : a \to C$ mapping the state $S_\alpha$ to its color in $a$. The sentence

$$\phi(S_\alpha, S_\beta) = (S_\alpha \neq S_\beta) \wedge \Big(\text{Color}(S_\alpha) \neq \text{Color}(S_\beta)\Big) \wedge (|C| \leq 4)$$

may be constructed for each pair of neighboring states $S_\alpha, S_\beta \in a$. Then

$$M_a = \bigcup_{\substack{S_\alpha \in a \\ S_\beta \in a \\ S_\alpha \text{ neighbors } S_\beta}} \phi(S_\alpha, S_\beta)$$

is a set of sentences which must all be true if the four-color property holds for the map $a$. If any one is false, the property does not hold for that map. By the compactness theorem there exists a finite subset $m_a$ of this set which sufficies to prove the theorem on this map. Let

$$M = \bigcup_a m_a.$$

This is a (possibly infinite) set of sentences. Therefore by the compactness theorem there exists a finite set $M_0$ that models $\phi$.

Then the set $M_0$ corresponds to a finite portion of the map $a$. Since this portion is all that is needed to prove the four-color property of this map, it may be considered independently as a finite map. Because this can be done for any colored map, only finite maps need be considered for the four-color theorem. $\boxed{\text{Q.E.D.}}$