Logic Homework 6

D. Dakota Blair

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Abstract

A presentation of solutions to the sixth homework.

Definitions

We call an atomic symbol a *token*. A set of tokens, namely A, is an *alphabet*. Then A combined with concatenation forms a monoid A^*. An element $a \in A^*$ is a *word*, in particular a word over A. Let the *empty word*, the neutral element with respect to concatenation, be ϵ. Let $|a|$ denote the length of $a \in A^*$ with $|\epsilon| = 0$. A word x is a *factor* of the word y if there exists words a, b such that $y = axb$. We call a a *prefix* of y and b a *suffix* of y. A factor x of y is *proper* when $x \neq y$ and $x \neq \epsilon$.

The *terms* of a signature σ are defined recursively as follows.

(i) All variables are terms.

(ii) All constant symbols are terms.

(iii) If $t_0, \ldots, t_{n-1}$ are terms and $f \in F$ with $\sigma'(f) = n$, then $f(t_0, \ldots, t_{n-1})$ is a term too.

(iv) Finally, t is a term if it can be built in finitely many steps using (i)-(iii).

Problem 2.2.3

(About unique legibility) Prove that no proper initial segment of a term (regarded as a string of symbols of the alphabet) can be a term. Derive that for every term there is a unique way of building it up from its constituents according to the above recursion.

Terms are words over the alphabet A consisting of tokens for variables, constant symbols, nonlogical symbols inherited from the signature, '(', ')', and ','.

Proof. From above, a given term $t = f(t_0, \ldots, t_{n-1})$ falls into one of three cases, either it is a variable token, a constant symbol or a term of type (iii). Since there are no proper prefixes of any term falling into the first two cases, we need not consider them further. Note that (iv) merely restricts terms to be finite in steps of construction and hence length as a word. This is because the terms of (iii) are themselves defined using terms rather than variables or constant symbols. Thus the only case to consider is that of (iii). Let us call the tokens '(' and ')' *parentheses* and when speaking of a factors '(' or ')' in t we shall call it a *parenthetical factor*. In order for a term to be well-formed the parentheses must be matched. That is, each factor '(' must correspond to one and only one factor equal to ')'.

It follows that the number of distinct parenthetical factors must be equal. Then the total number of parenthetical factors is even. Any proper prefix that contains all ')' factors of t except the suffix of length 1 has an odd amount of parenthetical factors, so it is not a well-formed term.

Further a factor '(' may only be matched to a ')' which occurs in a suffix not containing the first factor. In other words '(' must occur *before* its matched ')', likewise ')' must occur *after* its matched '('. A necessary condition for matching all parenthetical factors in a given term is that in particular the outermost '(' and ')' must be matched to each other.

Assume a proper prefix p of t is well-formed. Then the first '(' is matched to a factor ')' of the shortest suffix of p containing such a factor. Call this factor f. Note that all other relevant tokens are matched. In particular there is no factor '(', other than the first, that matches f. But in the full term t the suffix of length 1 is ')' hence it must match the first '('. This leaves f unmatched, and we noted earlier that there is no previous factor which matches f. Therefore we have found an unmatched ')' token, contradicting the assumption that p is well-formed.

The possibility that a different match could be defined which would circumvent this problem. But if that were the case then a prefix of t with f as the last token would be subject to the analysis in the paragraph above. Specifically, there are an equal number of parenthetical terms in t and taking this prefix removes a single ')' term. Then an unequal number of arenthetical terms remains, hence this prefix is not well-formed. Q.E.D.

Problem 2.2.1

Show that any map h_0 from X to an L-structure $\mathcal{M}$ can be uniquely extended to a homorphism h from $\mathrm{Term}_L(X)$ to $\mathcal{M}$.

Proof. We begin with the map h_0 mapping X into $\mathcal{M}$, that is $h_0(x_i) = m_i$ for $x_i \in X$, $m_i \in \mathcal{M}$. Then we define $h(x_i) = m_i$, $h(c) = c$ for all constants c in $\mathcal{M}$ and $h(f(x_{i_0}, \ldots, x_{i_{\sigma'(f)}})) = f(h(x_{i_0}), \ldots, h(x_{i_{\sigma'(f)}}))$ for all functions f of $\mathcal{M}$.

Uniqueness is assured, as if $h = g$ for two homomorphisms extended from h_0 they must agree on the value of all constants and variables. Thus they are only be allowed to differ on compound terms. However unique readability shows that any compound term $g(t)$ can only be composed from the finite process defining terms. By the homomorphism property, $g(t)$ will be a term that will be dependent only upon the definition of h_0. But since g and h agree on h_0 this shows that $g(t) = h(t)$ for every t, moreover $h = g$. Q.E.D.

Problem 2.2.2

Let h_0 and h be as above and let $t(\bar{x})$ be an L-term whose variables $\bar{x}$ are in X. Prove that $t^{\mathcal{M}}(h_0[\bar{x}]) = h(t(\bar{x}))$.

Proof. This proceeds in much the same way as the uniqueness argument above. In particular, we need to show that the term as a member of $\mathcal{M}$ will be the same as its homomorphic image from $\mathrm{Term}_L(X)$. The homomorphic property of h and unique readability of t allows us to write $h(t(\bar{x}))$ as a term in $h(\bar{x})$, which is precisely $h_0[\bar{x}]$.

Note $t(\bar{x})$ is either a variable, a constant or a function of terms. If it is a constant or a variable we are done as we may consider constants as nullary functions. Specifically, $t^{\mathcal{M}}(h_0[\bar{x}]) = h(t(\bar{x}))$ from the definition of h. If it is a function of terms, that is $t(\bar{x}) = f(t_0(\bar{x}), \ldots, t_n(\bar{x}))$, we may use the homomorphic property of h to show that $h(t(\bar{x})) = f(h(t_0(\bar{x})), \ldots, h(t_n(\bar{x})))$. We have from unique readability that this can be done in one and only one way, and from the finiteness condition of terms that this process will result, after at most a finite number of steps, in functions of terms of the form $h(t_\alpha(\bar{x}))$ which, as stated earlier, satisfy the statement. Therefore

$$t^{\mathcal{M}}(h_0[\bar{x}]) = h(t(\bar{x})).$$

Q.E.D.