MODEL THEORY II HOMEWORK 10

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20. *Every model of a countable $\aleph_1$-categorical theory is ω-homogeneous.*

Proof. By the Baldwin-Lachalan theorem T is ω-stable. Let $\overline{a}, \overline{b}, a_0 \in \mathcal{M}$ such that $\overline{a} \equiv_M \overline{b}$. If $\mathcal{M}$ is ω-saturated then $\text{tp}(a_0/\overline{a})$ is realized in $\mathcal{M}$ by some element b_0. Thus $\overline{a}a_0 \equiv \overline{b}b_0$. If $\mathcal{M}$ is uncountable then it is ω-saturated by categoricity. Therefore the only case remaining is when $\mathcal{M}$ is countable and not ω-saturated. In this case, $p = \text{tp}(a_0/\overline{a})$ is isolated. If p is not isolated then there exists a ϕ such that $p \cup \{\phi\}$ and $\{\neg\phi\}$ are distinct, consistent and not isolated. Then a binary tree argument shows that there are uncountably many types over countably many parameters, which contradicts ω-stability of T. Therefore $\mathcal{M}, \overline{a}, a_0)$ is a constructible model and therefore ω-homogeneous. □

21. *Show there is a natural injection of M into $S_1(B)$ for $M \subset B$. Identify M with its image and let $\overline{M}$ be its, topological, closure in $S_1(B)$. Show that $q \in S_1(B)$ is in $\overline{M}$ if and only if q is a coheir of $q{\restriction}M$.*

Proof. Let $f : M \to S_1(B)$ be defined as $f(m) = \text{tp}(m/B)$. Note that this map is well defined as $M \subset B \subset \mathbb{M}$. Note that if $m_1, m_2 \in M$ are distinct then $q_1 = \text{tp}(m_1/B) \neq q_2 = \text{tp}(m_2/B)$ since the formula $x = m_1$ is in q_1 but $x \neq m_1$ is in q_2. Therefore f is injective.

Let $\overline{M} = \overline{f(M)}$ in $S_1(B)$ and $q \in \overline{M}$. To show that q is a coheir of $q{\restriction}M$ we need to see that it is finitely realized in M, that is for each formula such that $q \models \varphi(x, \overline{b})$ there is an $m \in M$ such that $\mathbb{M} \models \varphi(m, \overline{b})$. Let $q \models \forall x\varphi(x, \overline{b})$, we need to show $\exists m \in M$ such that $\varphi(m, n) \in q$. Since $q \models \forall x\varphi(x, \overline{b})$ there exists a $b \in B$ such that $U = \langle\varphi(b, \overline{b})\rangle \subset q$. Since U is an open set and $q \in \overline{M}$ we have that $U \cap M$ is nonempty. Thus there is an $m \in M$ such that $\text{tp}(m/B) = f(m) \subset q$. Therefore $\varphi(m, n) \in q$. Hence q is a coheir of $q{\restriction}M$.

For the converse, again let $b \in B$ and $U = \langle\varphi(b, \overline{b})\rangle$ and assume $U \subset q$ and therefore $q \models \varphi(b, \overline{b})$. Then since q is a coheir of $q{\restriction}M$ there is an $m \in M$ such that $p = \langle\varphi(m, n)\rangle \subset q$. Because $m \in M$ and $M \subset B$ we have $q = p \cap \text{tp}(m/B)$ is nonempty. Therefore $q \subset q{\restriction}M = q \cap M \subset M$. Hence any open set contained in q intersects M and consequently $q \in \overline{M}$. □