MODEL THEORY II HOMEWORK 11

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22. *Prove that every ultrapower of a type p over M is an heir of p.*

Proof. Let U be a nonprincipal ultrafilter on I and M^U and p^U be the ultrapowers of M and p respectively. Let a be a realization in p such that $a \in N$, an elementary extension of M with ultrapower N^U. Let $g : M \to M^U$ and $f : N \to N^U$ denote the respective diagonal inclusion maps. Given $\mu \in M^U$ denote by $\pi_i(\mu)$ the value of the ith coordinate of μ. Then $\varphi(\overline{x}, \overline{y}) \in p^U$ implies $\exists b \in N$ such that $\varphi(\overline{x}, \alpha) \in f(p)$. This is true if and only if there exists an $A \in U$ and a $\mu \in M^U$ such that $\forall i \in A$ we have $M^U \models \varphi(\overline{x}, f(\pi_i(\mu)))$ Thus $\varphi(\overline{x}, \mu) \in p^U \Rightarrow \varphi(\overline{x}, \pi_i(\mu)) \in p$, and $\varphi(\overline{x}, f(\pi_i(\mu))) \in p^U$ $N \models \varphi(a, \pi_i(\mu)) \Rightarrow N^U \models \varphi(f(a), f(\pi_i(\mu)))$ hence $\varphi(\overline{x}, f(\pi_i(\mu))) \in f(p)$. Therefore p^U is an heir of $f(p)$. □

23. *Prove* $\mathrm{tp}(\overline{a}/M\overline{b})$ *is an heir of* $\mathrm{tp}(\overline{a}/M)$ *if and only if* $\mathrm{tp}(\overline{b}/M\overline{a})$ *is an coheir of* $\mathrm{tp}(\overline{b}/M)$

Proof. The type $\mathrm{tp}(\overline{a}/M\overline{b})$ is an heir of $\mathrm{tp}(\overline{a}/M)$ if and only if for all φ such that $\varphi(\overline{x}, \overline{m}, \overline{b}) \in \mathrm{tp}(\overline{a}/M\overline{b})$ there is an $\overline{m'} \in M$ such that $\varphi(\overline{x}, \overline{m}, \overline{m'}) \in \mathrm{tp}(\overline{a}/M\overline{b})$. This is true if and only if, using dummy variables as necessary for the converse, $\varphi(\overline{a}, \overline{m}, \overline{x}) \in \mathrm{tp}(\overline{b}/M\overline{a}) \Rightarrow \mathbb{M} \models \varphi(\overline{a}, \overline{m}, \overline{m'})$ which is true if and only if $\mathrm{tp}(\overline{b}/M\overline{a})$ is an coheir of $\mathrm{tp}(\overline{b}/M)$. □