# MODEL THEORY II HOMEWORK 3

**6** (Rothmaler 12.1.7)**.** *Suppose $\mathcal{M}$ is $\kappa$-saturated, $A \subset M$ with $|A| < \kappa$ and $n > 1$ is a natural number. Prove that all $n$-types of $\mathcal{M}$ over $A$ are realized in $\mathcal{M}$.*

*Proof.* Assume $|A| < \kappa$ and all $(n-1)$-types of $\mathcal{M}$ over $A$ are realized in $\mathcal{M}$. If $p \in S_n(A)$ then

$$p = \left\{ \varphi(\overline{v}) \middle| |\overline{v}| = n \text{ and } \mathcal{M} \models \varphi(\overline{v}) \right\}.$$

Let $\overline{v} = \{v_i\}_{i=1}^n$ and define

$$q = \left\{ \exists v_n \varphi(v_n) \middle| \varphi \in p \right\}.$$

Note that $q$ is a consistent set of $L_{n-1}$ formulas. If $\varphi \in L_{n-1}, \neg\varphi \notin q$ and $\varphi \notin q$ then neither $\exists v_n \varphi(v_n)$ nor $\exists v_n \neg\varphi(v_n)$ (which is logically equivalent to $\neg\exists v_n \varphi(v_n) \in q$) are in $p$. This contradicts the completeness of $p$. Therefore $q \in S_{n-1}(A)$ and is realized in $\mathcal{M}$ by, say, $m_q \in M^{n-1}$ according to the induction hypothesis. If $m_q$ realizes $p$ then the proof is finished. If $m \in M^n$ is not a realization of $p$ then there is a formula $\varphi(\overline{v}) \in p$ such that $\mathcal{M} \models \neg\varphi(m)$. Then define

$$p' = \left\{ \varphi(m, v) \in L_1 \middle| \varphi(\overline{v}) \in p \text{ and } |\overline{v}| = n \right\}.$$

If $\varphi \in L_1, \varphi(v_n) \notin p'$ and $\neg\varphi(v_n) \notin p'$ then $\varphi(m, v_n) \notin p$ and $\neg\varphi(m, v_n) \notin p$ which contradicts the completeness of $p$. Therefore $p'$ is complete, hence $p' \in S_1(A)$ is realized by $m_{p'} \in \mathcal{M}$ by the saturation of $\mathcal{M}$. Thus $m = m_q \cup m_{p'}$ is a realization of $p$, and the result follows. □

**7.** *If $\mathcal{M}$ is a homogeneous structure then every partial elementary map $f : \mathcal{M} \to \mathcal{M}$ where $|\mathrm{Dom}(f)| < |M|$ can be extended to an $F \in \mathrm{Aut}(\mathcal{M})$.*

*Proof.* Let $A = \mathrm{Dom}(f)$ and $B = f[\mathcal{M}]$. Let $|A| = \kappa < |\mathcal{M}|$. Then well-order $A$ and $B$ as $A = \{a_i\}_{i<\kappa}$ and $B = \{b_i = f(a_i)\}_{i<\kappa}$, respectively. This gives rise to an elementary chain $\{\mathcal{M}_i\}_{i<\kappa}$ such that

$$\mathcal{M}(A) = \mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_i \prec \cdots$$

By homogeneity $f$ can be extended by picking $a_{\kappa+1} \in M \setminus A$ and finding $b_{\kappa+1} \in M$ such that $\mathrm{tp}(a/A) = \mathrm{tp}(b/B)$.

Therefore by the elementary chain lemma $\mathcal{M}_{\kappa^+} = \cup_{i<\kappa^+} \mathcal{M}_i \prec \mathcal{M}$. This argument works for every $\kappa < \mathcal{M}$, and therefore if $\kappa$ is such that $\kappa^+ = |\mathcal{M}|$ we have that $\mathcal{M}_{\kappa^+} = \mathcal{M}$. Otherwise, one can form another elementary chain indexed by all $\kappa < \mathcal{M}$, and the union of these will be $\mathcal{M}$. Further, this model is defined as the image of $F$, the extension of $f$. Hence $F$ is a (total) elementary map such that $F(A) = f(A)$. Therefore $F \in \mathrm{Aut}(\mathcal{M})$ is an automorphism of $\mathcal{M}$ extending $f$. □