MODEL THEORY II HOMEWORK 4

9. *Let $(\mathcal{N}, \mathcal{M})$ be an elementary pair of L-structures with predicate P such that $\mathcal{N} \simeq \mathcal{M}$ and f a unary function symbol specifying the isomorphism. The statement "f is an isomorphism between $\mathcal{N}$ and $\mathcal{M}$" is a first-order statement.*

Proof. Since f is an isomorphism, it is injective, surjective and a homomorphism. These can be said as follows:

- Injective: $I_{x,y} = (f(x) = f(y) \to x = y)$
- Surjective: $S_y = (P(y) \to \exists x(f(x) = y))$.
- Homomorphism: $H_{x,y,g} = f(g(x, y)) = g(f(x), f(y))$.

Thus

$$T = \{\neg\exists(x, y)\neg Ix, y\} \cup \{\neg\exists y\neg S_y\} \cup \{\neg\exists(x, y)\neg H_{x,y,g} | g \text{ is a function symbol in } L\}$$

is a first-order axiomatization of the statement "f is an isomorphism between $\mathcal{N}$ and $\mathcal{M}$." □

Lemma 9.1 (Lemma 12 (i)). *Every countable theory T has a countable homogeneous model.*

Lemma 9.2 (Lemma 15). *Let L be countable. Then every elementary pair is elementarily quivalent to some $(\mathcal{N}, \mathcal{M})$ where $\mathcal{N} \simeq \mathcal{M}$, and both $\mathcal{N}$ and $\mathcal{M}$ are countable and homogeneous.*

10. *Check the ω-homogeneity of $\mathcal{M}''$ and $\mathcal{N}''$ in the last sentence of the proof of Lemma 15.*

Proof. Note that $T = \mathrm{Th}(\mathcal{N}, \mathcal{M})$ is a countable theory. In the proof, $(\mathcal{N}', \mathcal{M}')$ is an arbitrary elementary pair of models of T. The first conclusion of Lemma 12 ensures the existence of a homogeneous model of T In particular we have that $(\mathcal{N}'', \mathcal{M}'', f'') \equiv (\mathcal{N}', \mathcal{M}', f)$ is a countable, homogeneous elementary pair in $L(P'', f'')$, but homogeneity is preserved in reducts, in particular in L. Therefore $\mathcal{N}''$ is countable and homogenous, and $\mathcal{M}'' \preceq \mathcal{N}''$ is as well. □