MODEL THEORY II HOMEWORK 5

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11. *A countable theory T has a countable atomic model $\mathcal{N}$ that is not minimal if and only if T has an atomic model $\mathcal{M}$ of power $\aleph_1$.*

Proof. If $\mathcal{N}$ is not minimal, then there is a structure $\mathcal{M}$ such that $\mathcal{M} \simeq \mathcal{N}$ and $\mathcal{M} \prec \mathcal{N}$. Let f be an isomorphism from $\mathcal{N}$ to $\mathcal{M}$. Therefore f is an elementary map, so by isomorphic correction we can find $\mathcal{N}_1$ such that $N \prec N_1$ by interpreting $\mathcal{M}$ as a copy of $\mathcal{N} = \mathcal{N}_0$. Further since $\mathcal{N}_0$ is countable and atomic, it is homogeneous, therefore $\mathcal{N}_1$ is also atomic. This process forms a proper elementary chain

$$\mathcal{N}_0 \prec \mathcal{N}_1 \prec \cdots \prec \mathcal{N}_\alpha \prec \cdots$$

from which we can form

$$\mathcal{N}_{\aleph_1} = \bigcup_{\alpha<\aleph_1} \mathcal{N}_\alpha.$$

This is an $\aleph_1$ union of countable sets, hence $|\mathcal{M}_{\aleph_1}| = \aleph_1$. By homogeneity of $\mathcal{N}_\alpha$ for $\alpha < \aleph_1$, the model $\mathcal{M}_{\aleph_1}$ is constructible and therefore atomic.

Conversely assume $\mathcal{N}$ is atomic with $|\mathcal{N}| = \aleph_1$. Then by Ehrenfeucht's Omitting Types Theorem, there exists an $\mathcal{M}$ which is countable and atomic. Therefore $\mathcal{M}$ is prime, and hence elementarily embeddable in $\mathcal{N}$. Then $(\mathcal{N}, \mathcal{M})$ form an $(\aleph_0, \aleph_1)$ elementary pair. By Löwenheim-Skolem and Ehrenfeucht's theorem there exists an elementary pair $(\mathcal{N}', \mathcal{M}') \equiv (\mathcal{N}', \mathcal{M}')$ which is countable and atomic. Then $\mathcal{M}'$ and $\mathcal{N}'$ are countable and elementarily equivalent, even to $\mathcal{M}$, consequently they are isomorphic. Thus $\mathcal{N}'$ is a countable atomic model which is not minimal. ☐

12. *Prove $(\mathbb{Z}/4\mathbb{Z})^\omega$ is almost strongly minimal.*

Proof. Identify the elements of $\mathbb{Z}/4\mathbb{Z}$ with the elements of 4. The formula φ that says $2x \neq 0$ is strongly minimial. It consists of elements which are sequences with finitely many 1s and 3s. The only complete types this model realizes are $x = 0$, $x \neq 0$, $2x = 0$, $2x \neq 0$ and $4x = 0$. Thus φ cannot be split into an infinite coinfinite definable set. Further $\operatorname{acl} \varphi$ is everything, therefore the model is almost strongly minimal. ☐