MODEL THEORY II HOMEWORK 6

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13. *Prove the following are equivalent for an L-theory T:*

(1) *The quantifier Q_α may be eliminated in T_α for some $\alpha > 0$.*
(2) *The theory T has no Vaughtian Pairs.*
(3) *For each formula $\varphi(x, \overline{y}) \in L$ there exists a n_φ such that $T_\alpha \models \forall \overline{y} \left(\exists^{\geq n_\varphi} x \varphi(x, \overline{y}) \to Q_\alpha x \varphi(x, \overline{y})\right)$.*

Proof. ($1 \Rightarrow 3$) Consider the contrapositive. If there is no uniform bound n_φ for a formula $\varphi(x, y)$, then

$$\{\neg Q_\alpha x \varphi(x, y)\} \cup \left\{\exists^{\geq n} x \varphi(x, y) \middle| n < \omega\right\}$$

is finitely consistent, and hence by compactness consistent. Therefore Q_α may not be eliminated in T_α in this circumstance.

($3 \Rightarrow 1$) This follows trivially from the definition of Q_α.

($1 \Rightarrow 2$) Again proceed by proving the contrapositive. By Vaught's two-cardinal theorem, if T has a Vaughtian pair, it has an $(\aleph_1, \aleph_0)$ model, say $(\mathcal{N}_1, \mathcal{M}_0)$. By Löwenheim-Skolem we can find $\mathcal{N}_\alpha \equiv \mathcal{N}_1$ such that $|\mathcal{N}_\alpha| = \aleph_\alpha$ and $\mathcal{M}_0 \subset \mathcal{N}_\alpha$, so using isomorphic correction we may construct an $(\aleph_\alpha, \aleph_0)$ model of T. Let $(\mathcal{N}, \mathcal{M})$ be an $(\aleph_\alpha, \aleph_0)$ Vaughtian pair such that $(\mathcal{N}, \mathcal{M}) \models T$. Then there exists a formula φ such that $\varphi(\mathcal{M})$ is infinite and $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$. Therefore $\varphi(\mathcal{M}) \subset \mathcal{M}$ and is hence countable, so $\aleph_\alpha$ may not be eliminated.

($2 \Rightarrow 3$) This follows from the proof of Lemma 20(i) by replacing countable with $\aleph_\alpha$ and uncountable by $> \aleph_\alpha$, again proceeding by the contrapositive. If there are no uniform bounds, then one can construct a structure for any $n < \omega$, say $\mathcal{N}_n$, such that $\mathcal{N}_n \models T$ and there exists $\varphi(x, \overline{y}), \overline{a_n} \in \mathcal{N}_n$ such that $n < |\varphi(\mathcal{N}_n, \overline{a_n})| < \omega$. Further one may take $|\mathcal{N}_n| > \aleph_\alpha$ as if the original choice is not large enough, by Löwenheim-Skolem we may take an elementary extension $\mathcal{N}'_n$ where $|\mathcal{N}_n| > \aleph_\alpha$ and $\varphi(\mathcal{N}_n, \overline{a_n}) = \varphi(\mathcal{N}'_n, \overline{a_n})$. Now take $\mathcal{M}_n$ such that $\varphi(\mathcal{N}_n, \overline{a_n}) \cup \overline{a_n} \subset M_n$, $|\mathcal{M}_n| = \aleph_\alpha$ and $\mathcal{M}_n \preceq \mathcal{N}_n$. Then $\mathcal{M}_n$ and $\mathcal{N}_n$ form an elementary chain, so define $\mathcal{M} = \cup_{n<\omega} \mathcal{M}_n$ and $\mathcal{N} = \cup_{n<\omega} \mathcal{N}_n$. Therefore $(\mathcal{N}, \mathcal{M})$ in $L(P)$ forms an elementary pair. But here we may construct

$$\begin{aligned} T_P = &\{\exists \overline{y}(\forall x(\varphi(x,\overline{c}) \to P(x)) \wedge P(\overline{c}))\} \cup \{\text{Tarski-Vaught for } P(\mathcal{N}) \preceq \mathcal{N}\} \\ &\cup \{\forall \overline{y}((P(\overline{y}) \wedge \exists x \psi(x,\overline{y})) \to \exists x P(x) \wedge \psi(x,\overline{y}))\}. \end{aligned}$$

Hence if $\overline{a}$ is a realization of T_P then $\mathcal{M} \preceq \mathcal{N}$ and $\varphi(\mathcal{N},\overline{a}) \subset M$. Therefore we may consider the theory

$$T'_P = T_P \cup \{\exists^{\geq n} x \varphi(x,\overline{c}) | n < \omega\}$$

which is finitely consistent since $(\mathcal{N}_n, \mathcal{M}_n, \overline{a_n}) \models T'_P$. But every model of this theory is a Vaughtian pair. Therefore if a theory has no uniform bounds then it has Vaughtian pairs. Hence no Vaughtian pairs implies that the theory admits uniform bounds. □

14. *If Q_1 may be eliminated in a complete theory T, then so is Q_0.*

Proof. Since Q_1 may be eliminated, T has uniform bounds. Then T has no Vaughtian pairs and hence no 2-cardinal models. Thus any formula $\varphi(x)$ such that $|\varphi(\mathcal{M})| \geq \aleph_0$ is consistent with $Q_\alpha x \varphi(x)$. Hence $|\varphi(\mathcal{M})| = \aleph_\alpha$. Therefore the same uniform bound that implies $Q_\alpha x \varphi(x)$ implies $Q_0 x \varphi(x)$. Hence Q_0 may be eliminated in T. □