MODEL THEORY II HOMEWORK 7

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16 (**)**.** *Let T be a countable, $\aleph_1$-categorical theory such that $\mathcal{N}$ and $\mathcal{M}$ are models where $\mathcal{N} \models T$ and $\mathcal{M} \models T$. Further let $\varphi(x) \in L(\mathcal{M}_0)$ be a strongly minimal formula over the prime model $\mathcal{M}_0$ of T. If $\mathcal{M} \preceq \mathcal{N}$ then* $\dim \mathcal{N} = \dim \mathcal{M} + \dim \mathcal{M}/\mathcal{N}$

Lemma (*)**.** *Let I be a basis of $\varphi(\mathcal{M})$, and J an independent set in $\varphi(\mathcal{N}) \setminus \mathcal{M}$. If $c \in \varphi(\mathcal{N})$ then $c \in \operatorname{acl} IJ$.*

Definition (p^A_φ)**.** *Let φ be a nonalgebraic formula over A. Then*

$$p^A_\varphi = \left\{ \varphi(x) \wedge \neg\psi(x) \middle| \psi \in L_1(A) \text{ and } |\psi(\mathcal{M})| < \omega \right\}$$

is a type that says, "$\varphi(x)$ and $x \notin \operatorname{acl} A$".

Remark. *The formula φ is strongly minimal if and only if p^A_φ is the unique nonalgebraic type in $S_1(A)$ containing φ.*

Proof of (*). Proceed by induction on $|J|$. If $|J| = 0$ then $\mathcal{N} = \mathcal{M}$ and therefore $c \in \varphi(\mathcal{M}) \subset \operatorname{acl} I$.

It is also important to consider another base case, in particular, the case when $J = \{b\}$ (so $|J| = 1$). By Lemma 22(4) M is atomic over Ib, that is for each $\overline{m} \in M$ the type $\operatorname{tp}(\overline{m}/Ib)$ is isolated. Since $c \in \varphi(\mathcal{N})$ there is an $\overline{m} \in M$ such that $c \in \operatorname{acl} I\overline{m}b$. Therefore $\operatorname{tp}(\overline{m}/Ib)$ is isolated, $\operatorname{tp}(c/I\overline{m}b)$ is isolated, and hence, by transitivity of isolation, $\operatorname{tp}(c/Ib)$ is isolated. But $\varphi(\mathcal{M}) \subset \operatorname{acl} I \subset \operatorname{acl} Ib$ and $\varphi(\mathcal{M})$ is infinite. Therefore by Lemma 22(1) the type p^{Ib}_φ is not isolated hence $c \not\models p^{Ib}_\varphi$, that is c is algebraic over Ib. Thus $c \in \varphi(\mathcal{N})$ implies $c \in \operatorname{acl} Ib$. Consequently if $|J| = 1$ and $c \in \varphi(\mathcal{N})$ then $c \in \operatorname{acl} IJ$.

The induction hypothesis is therefore that if $c \in \varphi(\mathcal{N})$ then $c \in \operatorname{acl} IJ$ for any $|J| \leq \alpha$.

Now consider the successor case where $J_{\alpha+1} = J_\alpha \cup \{b\}$ and $b \notin J_\alpha$. Take $\mathcal{N}'$ prime over $\mathcal{M}J_\alpha$. Note that $\dim \mathcal{N}/\mathcal{N}' = 1$, for $\varphi(\mathcal{N}) \subset \operatorname{acl} MJ_\alpha b$, and $MJ_\alpha \subset N'$, so $\varphi \subset \operatorname{acl} N'b = \operatorname{acl}_N b$. Therefore

$$\dim \mathcal{N} = \dim \mathcal{N}' + \dim \mathcal{N}/\mathcal{N}' = \dim \mathcal{N}' + 1.$$

By the induction hypothesis and Proposition 22(3)

$$\dim \mathcal{N}' = \dim \mathcal{M} + \dim \mathcal{N}'/\mathcal{M} = \dim \mathcal{M} + |J_\alpha|.$$

Thus

$$\dim \mathcal{N} = \dim \mathcal{M} + |J_\alpha| + 1 = \dim \mathcal{M} + |J_{\alpha+1}|.$$

Thus if the induction hypothesis holds for $|J| = \alpha$ it holds for $|J| = \alpha + 1$.

Now consider the limit case where the induction hypothesis holds for all $\alpha < \beta$ and β is a limit ordinal. Then for any $J' \subset J$ let $\mathcal{N}_{J'}$ be prime over MJ'. If $|J'| < |J|$ we have that $c \in \varphi(\mathcal{N}_{J'})$ implies that $c \in \operatorname{acl} IJ'$. If $c \in \varphi(\mathcal{N})$ then in particular there is a J' as above such that $c \in \varphi(\mathcal{N}_{J'})$. Otherwise

$$c \notin \bigcup_{|J'|<|J|} \mathcal{N}_{J'} \quad \text{but} \quad \mathcal{N} = \bigcup_{\alpha<|J|} \mathcal{N}_{J_\alpha}$$

where $J = \{b_i\}_{i \le |J|}$ and $J_\alpha = \{b_i\}_{i \le \alpha}$. Thus if $c \notin \mathcal{N}_{J_\alpha}$ for every $\alpha < |J|$ contradicts $c \in \varphi(N)$. By the induction hypothesis this shows $c \in \operatorname{acl} IJ'$. But $J' \subset J$ hence $\operatorname{acl} IJ' \subset \operatorname{acl} IJ$ implies $c \in \operatorname{acl} IJ$. Thus $c \in \varphi(\mathcal{N})$ implies $c \in \operatorname{acl} IJ$. Further we may choose J' large enough such that $\dim \mathcal{N}/\mathcal{N}_{J'} < \dim \mathcal{N}$. Hence $\dim \mathcal{N} = \dim \mathcal{N}_{J'} + \dim \mathcal{N}/\mathcal{N}_{J'}$ by the induction hypothesis of $(**)$. □

17. $I(T_E^{fcp}, \aleph_1) = \aleph_0$

Proof sketch. Each isomorphism class is uniquely identified by the cardinality of $\aleph_0$ and $\aleph_1$ classes. There are only $\aleph_0$ choices for these parameters in $\aleph_1$ models of T_E^{fcp}. Therefore $I(T_E^{\mathrm{fcp}}, \aleph_1) = \aleph_0$. □

18. *Let T be a countable $\aleph_1$-categorical theory. Show that $\mathcal{M} \models T$ is atomic and ω-saturated if and only if $\dim \varphi(\mathcal{M})$ is infinite for some (all) strongly minimal φ over $\mathcal{M}$.*

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