MODEL THEORY II HOMEWORK 7

DAKOTA BLAIR

Theorem (22). *Let T be a countable $\aleph_1$-categorical theory. If $\mathcal{M} \preccurlyeq \mathcal{N}$ then* $\dim \mathcal{N} = \dim \mathcal{M} + \dim(\mathcal{N}/\mathcal{M})$.

Proof. Let $\mathcal{M}_0$ be the prime model of T, $\varphi(x) \in L(\mathcal{M}_0)$ a strongly minimal formula, I a basis for $\varphi(\mathcal{M})$, J a maximally independent set in $\varphi(\mathcal{N}) \setminus \mathcal{M}$.

... Then from Proposition 22 $\dim(\mathcal{N}/\mathcal{M}) = |J|$. □

16. *Justify Theorem 22 for* $|J| \geq \aleph_0$.

Proof. □

17. *Calculate $I(T_E^{fcp}, \aleph_1)$.*

Proof that $I(T_E^{fcp}, \aleph_1) = \aleph_0$. Any $\aleph_1$ model of T_E^{fcp} must have at least one infinite congruence class. Further any infinite congruence class must have cardinality either $\aleph_0$ or $\aleph_1$. If there are no congruence classes of size $\aleph_1$ then there must be $\aleph_1$ congruence classes of cardinality $\aleph_0$. This defines a single isomorphism class, say A_0, and is characterized by having no $\aleph_1$ congruence classses and $\aleph_1$ many $\aleph_0$ classes.

Let $m = \aleph_1$ or $m \in \omega + 1$ such that $m > 0$. Similarly let $n = \aleph_1$ or $n \in \omega + 1$ such that $n > 0$. Note then there are $|\aleph_0 \cdot \aleph_0| = \aleph_0$ choices for (m, n). There is exactly one corresponding isomorphism class corresponding to a model with exactly n distinct $\aleph_0$ classes and exactly m distinct $\aleph_1$ classes. Assume two models $\mathcal{M}_1$ and $\mathcal{M}_2$ correspond to the same pair (m, n). Then any partial elementary map f preserving the finite equivalence classes may be extended to an isomorphism. Let the $\aleph_0$ and $\aleph_1$ congruence classes of $\mathcal{M}_1$ be $\{a_i\}_{i \in {}^1I_0}$ and $\{a_i\}_{i \in {}^1I_1}$ respectively. Similarly let the $\aleph_0$ and $\aleph_1$ congruence classes of $\mathcal{M}_2$ be $\{b_i\}_{i \in {}^2I_0}$ and $\{b_i\}_{i \in {}^2I_1}$ respectively. Define ${}^1I = {}^1I_0 \cup {}^1I_1$ and ${}^2I = {}^2I_0 \cup {}^2I_1$. Further define $g_1 : \mathcal{M}_1 \to {}^1I$ be the function which gives the index in ${}^1I$ of the congruence class of $\mathcal{M}_1$ containing the given element. Define $g_2 : \mathcal{M}_2 \to {}^2I$ similarly. Then $F : \mathcal{M}_1 \to \mathcal{M}_2$ will define an isomorphism when the following condition is satisfied: if $m \in \mathcal{M}_1, \alpha \in \{0, 1\}$ and $g_1(m) \in {}^1I_\alpha$ then $F(m) = f(m)$ or $g_2(F(m)) = f_\alpha(g_1(m))$.

Note that the isomorphism class of each such pair is distinct from A_0. Further two models $\mathcal{M}_1$ and $\mathcal{M}_2$ such that their corresponding pairs are distinct, that is $(m_1, n_1) \neq (m_2, n_2)$, define two distinct isomorphism classes. This is because any infinite congruence class must map to a class of the same cardinality. But if either $m_1 \neq m_2$ or $n_1 \neq_2$ then $|{}^1I_0| \neq |{}^2I_0|$ or $|{}^1I_1| \neq |{}^2I_1|$ which prevents the construction of an elementary bijection from $\mathcal{M}_1$ to $\mathcal{M}_2$. Then each pair (m, n) corresponds to a unique isomorphism class, say $A_{(m,n)}$.

Therefore A_0 and $A_{(m,n)}$ define $\aleph_0$ distinct isomorphism classes. These classes exhaust all $\aleph_1$ models of T_E^{fcp} as any other models would have a different cardinality. Hence $I(T_E^{\text{fcp}}, \aleph_1) = \aleph_0$. □

18. *Show that $\mathcal{M}$ as in Lemma 23 (1) is ω-saturated if and only if* $\dim \varphi(\mathcal{M})$ *is infinite for some (all) strongly minimal φ over $\mathcal{M}$.*

19 (optional)**.** *Explore ω-stable T where Lemma 23 holds*

Definition. *A M model is ω-**homogeneous** if for all $\overline{a} \equiv_M \overline{b}$ and a_0 there exists a b_0 such that $\overline{a}a_0 \equiv_M \overline{b}b_0$.*

Theorem (Baldwin-Lachlan)**.** *A countable theory T is κ-categorical if and only if T is ω-stable without Vaughtian Pairs.*

20. *Every model of a countable $\aleph_1$-categorical theory is ω-homogeneous.*

Proof. By the Baldwin-Lachalan theorem T is ω-stable. Let $\overline{a}, \overline{b}, a_0 \in \mathcal{M}$ such that $\overline{a} \equiv_M \overline{b}$. If $\mathcal{M}$ is ω-saturated then $\text{tp}(a_0/\overline{a})$ is realized in $\mathcal{M}$ by some element b_0. Thus $\overline{a}a_0 \equiv \overline{b}b_0$. If $\mathcal{M}$ is uncountable then it is ω-saturated by categoricity. Therefore the only case remaining is when $\mathcal{M}$ is countable and not ω-saturated. In this case,

□

21. *Show there is a natural injection of M into $S_1(B)$ for $M \subset B$. Identify M with its image and let $\overline{M}$ be its, topological, closure in $S_1(B)$. Show that $q \in S_1(B)$ is in $\overline{M}$ if and only if q is a coheir of $q|_M$.*

Proof. Let $f : M \to S_1(B)$ be defined as $f(m) = \text{tp}(m/B)$. Note that this map is well defined as $M \subset B \subset \mathbb{M}$. Note that if $m_1, m_2 \in M$ are distinct then $q_1 = \text{tp}(m_1/B) \neq q_2 = \text{tp}(m_2/B)$ since the formula $x = m_1$ is in q_1 but $x \neq m_1$ is in q_2. Therefore f is injective.

Let $\overline{M} = \overline{f(M)}$ in $S_1(B)$ and $q \in \overline{M}$. To show that q is a coheir of $q|_M$ we need to see that it is finitely realized in M, that is for each formula $\varphi(x, \bar{b}) \in q$ there is an $m \in M$ such that $\mathbb{M} \models \varphi(m, \bar{b})$. Since q is a complete type it is consistent and therefore finitely consistent.

If q is a coheir of $q|_M$ then for each $\varphi(x, \bar{b}) \in q$ there is an $m \in M$ such that $\mathbb{M} \models \varphi(m, \bar{b})$. $\square$

22. *Prove that every ultrapower of a type p over M is an heir of p.*

Proof. Let U be a nonprincipal ultrafilter on I and M^U and p^U be the ultrapowers of M and p respectively.

$$(p)_M = \cup_{m<\omega}\{\varphi(\overline{x},\overline{y}) \in L_{n+m} | \exists \overline{a} \in A^m \text{ such that } \varphi(\overline{x},\overline{a}) \in p\}$$

p^U is an heir of p if $(p^U)_M = (p)_M$

$$(p^U)_M = \cup_{m<\omega}\{\varphi(\overline{x},\overline{y}) \in L_{n+m} | \exists \overline{a} \in A^m \text{ such that } \varphi(\overline{x},\overline{a}) \in p^U\}$$

Let $\overline{a}$ be some realization of p in N where $M \prec N$. then $p^U = \text{tp}((\overline{a}^I/U)/M^U)$
$M \to M^U$ $\overline{a} \mapsto \overline{a}^U$

□

23. *Prove* $\text{tp}(\overline{a}/M\overline{b})$ *is an heir of* $\text{tp}(\overline{a}/M)$ *if and only if* $\text{tp}(\overline{b}/M\overline{a})$ *is an coheir of* $\text{tp}(\overline{b}/M)$

24. *Let T_E^∞ be the theory of an equivalence relation with infinitely many infinite congruence classes.*

a) Analyze the complete types over $A \subset M \models T_E^\infty$, especially $A = M$.
b) Analyze the definable complete types over $A \subset M \models T_E^\infty$.
c) Let $M \prec N$ and $p \in S(M), q \in S(N), p \subset q$. Find when q is an heir.
d) Let $M \prec N$ and $p \in S(M), q \in S(N), p \subset q$. Find when q is an coheir.

Definition. *Let $\mathcal{M} \prec \mathcal{N}$ and $q \in S(N)$. Say q is **$\mathcal{M}$-invariant** if $\forall f \in \text{Aut}(\mathcal{N}/\mathcal{M})$ then $q \subset f(q)$.*

25. *a) If q does not split over $\mathcal{M}$ then q is $\mathcal{M}$-invariant.*
b) if $\mathcal{N}$ is strongly homogeneous over $\mathcal{M}$ and q is $\mathcal{M}$-invariant then q does not split over $\mathcal{M}$.
c) In particular q is a global type over a model if and only if it is $\mathcal{M}$-invariant.

26. *Let $\mathcal{M} \models T, p \in S(\mathcal{M}), q \in S(\mathbb{M})$.*

a) q is an heir of $q|_M$.
b) q is definable over M.
c) q is $\mathcal{M}$-invariant
d) q is a coheir of $q|_M$.

For $\eta \models DLO_{--}$ find which implications fail. Which implications are true for any T?

Proof. $(b \Rightarrow c)$ is always true. If q is definable over M then it does not split over M by lemma 28(4). If q does not split over M then it is $\mathcal{M}$-invariant by (HW.25.a).

□