INTERPRETABILITY

DAKOTA BLAIR

Definition 0.1 (Interpretation)**.** *Let L and K be languages. Let $\mathcal{M}$ be an L-structure and $\mathcal{N}$ be a K-structure. If $\mathcal{M}^*$ is an L^*-structure, where $K \subset L^*$ such that there is a map $\cdot\,^* : L \to L^*$ such that $\mathcal{M} \models \varphi \iff \mathcal{M}^* \models \varphi^*$ then $\mathcal{M}^*$ and every structure isomorphic to it is an **interpretation** of $\mathcal{M}$ in $\mathcal{N}$.*

Remark 0.1. *By the interpretation lemma, to define the map $\cdot^*$ it suffices to show that every constant, function and relation of L is definable in K.*

Example 0.2. *Interpreting $\mathbb{C} = (\mathbb{C}, 0, 1, +, -, \cdot)$ in $\mathbb{R}$ as a $(\mathbb{R}, 0, 1, +, -, \cdot)$ structure.*

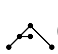
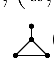
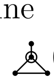
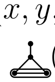
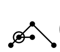
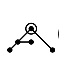
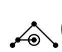
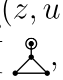
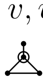
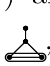
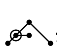
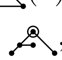
Given $x \in \mathbb{R}$ denote by x^{-1} the unique real number y such that $xy = 1$. Note that addition $(+(z, u, v))$, 0, multiplication $(\cdot(z, u, v))$, 1 and inverses are definable, in particular, if $0^* = (0, 0)$ and $1^* = (1, 0)$ then

$$\begin{aligned}
+(z,u,v) &\overset{*}{\longmapsto} \varsigma(x,y,a,b,c,d) \iff (x = a + c) \wedge (y = b + d)\\
\cdot(z,u,v) &\overset{*}{\longmapsto} \mu(x,y,a,b,c,d) \iff (x = ac - bd) \wedge (y = ad + bc)\\
\exists^{=1} z'(zz' = 1) &\overset{*}{\longmapsto} \iota(x,y,a,b) \iff \exists^{=1} xy(x = a(aa + bb)^{-1}) \wedge (y = -b(aa + bb)^{-1}).
\end{aligned}$$

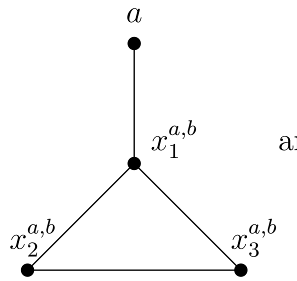
We therefore may use the following abbreviations for these operations, understanding (a, b) to be $a + bi$:

$$\begin{aligned}
(a,b) + (c,d) &= (a + c, b + d)\\
(a,b)(c,d) &= (ac - bd, ad + bc)\\
(a,b)^{-1} &= (a(aa + bb)^{-1}, -b(aa + bb)^{-1}).
\end{aligned}$$

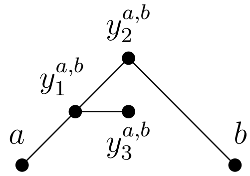
Example 0.3. *Interpreting linear orders in graphs as $\{R\}$-structures.*

Let (x, y, u, v, w) be the formula asserting that x, y, u, v and w are distinct, there are edges $(x, u), (u, v), (u, w)$ and (v, y) and these are the only edges involving vertices u, v and w. Define (x, y, u, v) in a similar fashion. Now define (z) as $\exists uvw$ (z, u, v, w) and (z), (z), (z), (z), (z) analagously. Finally, let $S = \{$ , , , , ,  $\}$.

If $(A,<)$ is a linear order then define a graph G_A such that for each $(a,b) \in A^2$



and (if $a<b$)



are subgraphs of G_A. Further construct G_A so that these are the only edges which involve any x_i^a or $y_i^{a,b}$.

Lemma 0.4. *If $(A,<)$ is a linear order then for all verticies $x \in G_A$ there is exactly one $\varphi(x) \in S$ such that $G_A \models \varphi(x)$.*

Let T be the theory in the language of graphs with the following axioms:

(1) R is symmetric and irreflexive
(2) Exactly one $\varphi \in S$ is true for every x.
(3) If (x) and (y) then $\neg R(x,y)$.
(4) If $\exists uvw\,(x,y,u,v,w)$ then $\forall u_1v_1w_1 \neg\,(y,x,u_1,v_1,w_1)$.
(5) If $\exists uvw\,(x,y,u,v,w)$ and $\exists u_1v_1w_1\,(y,z,u_1,v_1,w_1)$ then $\exists u_2v_2w_2\,(x,z,u_2,v_2,w_2)$.
(6) If (x) and (y) then either $x=y$, $\exists uvw\,(x,y,u,v,w)$ or $\exists uvw\,(y,x,u,v,w)$
(7) If $(x,u,v,w) \wedge (x,u_1,v_1,w_1)$ then $u=u_1$ and $v=v_1 \wedge w=w_1$ or $v=w_1 \wedge w=v_1$.
(8) If $(x,y,u,v,w) \wedge (x,y,u_1,v_1,w_1)$ then $u=u_1, v=v_1$ and $w=w_1$.

Remark 0.5. *If $(A,<)$ is a linear order then $G_A \models T$.*

Remark 0.6. *We may also interpret other common order-related axioms in this language. Using the interpretation map above we see that density ($\forall xy(x<y) \rightarrow \exists z(x<z \wedge z<y)$) becomes:*

$$\forall xy\,(x) \wedge (y) \wedge \exists uvw\,(x,y,u,v,w) \rightarrow$$
$$\exists z\,(z) \wedge$$
$$\exists u_x v_x w_x\,(u_x) \wedge (v_x) \wedge (w_x) \wedge (x,z,u_x,v_x,w_x) \wedge$$
$$\exists u_y v_y w_y\,(u_y) \wedge (v_y) \wedge (w_y) \wedge (z,y,u_y,v_y,w_y).$$