

# ARITHMETIC COMBINATORICS HOMEWORK

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Let  $\mathbb{P}$  be the set of primes.

1. Prove that for four pairwise tangent circles their respective curvatures are related by

$$2(a_1^2 + a_2^2 + a_3^2 + a_4^2) = (a_1 + a_2 + a_3 + a_4)^2.$$

*Proof.* Let the circles be centered at  $x_i$ , with radii  $r_i$  and curvature  $a_i$ . Without losing generality let  $x_1 = (0, 0)$ . Let  $c_i$  be the vectors connecting  $x_1$  to  $x_i$ . This results in 3 nonzero vectors in  $\mathbb{R}^2$  therefore they are linearly dependent, that is

$$\det((x_i x_j)) = \begin{vmatrix} x_2 \cdot x_2 & x_2 \cdot x_3 & x_2 \cdot x_4 \\ x_3 \cdot x_2 & x_3 \cdot x_3 & x_3 \cdot x_4 \\ x_4 \cdot x_2 & x_4 \cdot x_3 & x_4 \cdot x_4 \end{vmatrix} = 0.$$

But

$$\det((x_i x_j)) = \frac{4}{a_1 a_2 a_3 a_4} \left( \left( \sum a_i \right)^2 - 2 \sum a_i^2 \right) = 0$$

which is true if and only if

$$\left( \sum a_i \right)^2 = 2 \sum a_i^2$$

as was claimed. □