ARITHMETIC COMBINATORICS HOMEWORK 3

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Let $e(x) = e^{2\pi i x}$.

1. *Let* $\gamma \in SL_2(\mathbb{R})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $z \in \mathbb{H}$. *Show*

$$\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2}$$

Proof. Let $z = u + iv$ with $\Im(z) = v$. Then

$$\begin{aligned}
\Im(\gamma z) &= \Im\left(\frac{az+b}{cz+d}\right) = \Im\left(\frac{au+b+aiv}{cu+d+civ}\right) \\
&= \Im\left(\frac{(au+b+aiv)(cu+d-civ)}{|cz+d|^2}\right) \\
&= \Im\left(\frac{i(ad-bc)v + ac(u^2+v^2) + (ad+bc)u + bd}{|cz+d|^2}\right) \\
&= \frac{v}{|cz+d|^2} \\
\Im(\gamma z) &= \frac{\Im(z)}{|cz+d|^2}.
\end{aligned}$$

Thus $\mathbb{H}$ is invariant under this action of $SL_2(\mathbb{R})$. □

2. *If f and f' are modular forms of weight $2k$ and $2k'$ respectively, then ff' is a modular form of weight $2k+2k'$.*

Proof. The additivity of the weights follows directly from the definition of $e(z)$. If f and f' are holomorphic at ∞ then they are holomorphic in a neighborhood of infinity, that is, a region of the form $\left\{z \in \hat{\mathbb{C}} \middle| |z| > R\right\}$. Since the product of two holomorphic functions is holomorphic, ff' is also holomorphic in this region, and is therefore holomorphic at ∞. □

3. *Let* $\gamma \in SL_2(\mathbb{R})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Show* $\frac{d\gamma}{dz} = (cz+d)^{-2}$ *and*

$$f(z) = (cz+d)^{-2k} f(\gamma z) \iff \frac{f(\gamma z)}{f(z)} = \left(\frac{d\gamma}{dz}\right)^{-k} \iff f(\gamma z) d\gamma^k = f(z) dz^k.$$

Proof. Note

$$\frac{d\gamma}{dz} = \frac{(cz+d)a - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = (cz+d)^{-2}.$$

The rest follows immediately upon the observation that

$$\left(\frac{d\gamma}{dz}\right)^{-k} = (cz+d)^{2k}.$$

□

4. *Let* $G = (SL_2(\mathbb{Z})/\langle \pm 1\rangle)$, $\gamma \in SL_2(\mathbb{Z})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $w_1, w_2 \in \mathbb{C}, \Im \frac{w_1}{w_2} > 0$ *and* $\gamma(w_1, w_2) = (w_1', w_2') = (aw_1 + bw_2, cw_1 + dw_2)$.

(i) Show $\gamma(w_1, w_2)$ *is a basis for* $\Gamma(w_1, w_2)$.

(ii) Note that this implies $\Im \frac{w_1'}{w_2'} > 0$ *and hence they are a basis for* $\Gamma(w_1', w_2')$.

(iii) Show $\alpha = (w_1, w_2)$ *and* $\beta = (w_1', w_2')$ *define the same lattice if and only if there is a* $\gamma \in SL_2(\mathbb{Z})$ *such that* $a = \gamma b$.

(iv) Define $\mu((w_1, w_2)) = w_1'/w_2'$. *Show that* μ *gives a bijection from all lattices to* $\mathbb{H}/G$.

Proof. By assumption $\Im(\frac{w_1}{w_2}) > 0$, therefore w_1 and w_2 are linearly independent in $\mathbb{C}$ as a real vectorspace. The second statement follows from the fact that $SL_2(\mathbb{Z})$ is a group and therefore γ has an inverse. Therefore the system $\gamma\alpha = \beta$ can be solved. Thus $\Im(\mu(\alpha)) > 0$ is true if and only if $\Im(\mu(\beta)) > 0$. Further, for the unproven direction of the third statement, if α and β define the same lattice then their components may be written as integer linear combinations of one another and the corresponding matrix gives the necessary γ. Finally, to note that μ is well-defined it is necessary only to show that factoring by $\mathbb{C}^\times$ commutes with factoring by G. □