ARITHMETIC COMBINATORICS HOMEWORK

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Let $\mathbb{P}$ be the set of primes.

1.

a) Use Abel Summation to estimate

$$\sum_{n\le x} \log n \tag{$*$}$$

b) Use Euler-Maclaurin to estimate and compare.
c) Estimate

$$\sum_{2\le p\le x} \log p$$

d) See how Stirling's Formula (below) compares with (a) and (b).

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

e) Compute and estimate

$$\int_1^\infty \frac{\operatorname{frac}(t)}{t^2}dt \quad \textit{and} \quad \int_1^\infty \frac{\operatorname{frac}(t)}{t}dt$$

Estimate of $(*)$ using Abel Summation.

$$\begin{aligned}
\sum_{n\le x} \log n &= \lfloor x\rfloor \log x - \int_1^x \frac{\lfloor t\rfloor}{t}dt \\
&= \lfloor x\rfloor \log x - \int_1^x \frac{t - \operatorname{frac} t}{t}dt \\
&= \lfloor x\rfloor \log x - \int_1^x dt + \int_1^x \frac{\operatorname{frac} t}{t}dt \\
&= \lfloor x\rfloor \log x - (x-1) + \int_1^x \frac{\operatorname{frac} t}{t}dt \\
\sum_{n\le x} \log n &= \lfloor x\rfloor \log x - (x-1) + O(\log x)
\end{aligned}$$

□

Theorem (Euler-Maclaurin Summation)**.**

$$\sum_{k=1}^{n} f(k) = \int_1^x f(t)dt + \int_1^x \operatorname{frac}(t)f'(t)dt - f(x)\operatorname{frac}(x) \tag{EM}$$

Estimate of (∗) using Euler-Maclaurin summation. Using (EM) with $f = \log n$

$$\begin{aligned}
\sum_1^n \log(n) &= \int_1^x \log(t)dt + \int_1^x \frac{\operatorname{frac}(t)}{t}dt - \log(x)\operatorname{frac}(x) \\
&= x\log x - x + 1 + \int_1^x \frac{\operatorname{frac}(t)}{t}dt - \log(x)\operatorname{frac}(x) \\
&= \lfloor x \rfloor \log x - x + 1 + \int_1^x \frac{\operatorname{frac}(t)}{t}dt \\
\sum_1^n \log(n) &= \lfloor x \rfloor \log x - x + 1 + O(\log x)
\end{aligned}$$

□

Estimate of (∗) using Abel Summation.

$$\begin{aligned}
\sum_{n \le x} \log n &= \lfloor x \rfloor \log x - \int_1^x \frac{\lfloor t \rfloor}{t}dt \\
&= \lfloor x \rfloor \log x - \int_1^x \frac{t - \operatorname{frac} t}{t}dt \\
&= \lfloor x \rfloor \log x - \int_1^x dt + \int_1^x \frac{\operatorname{frac} t}{t}dt \\
&= \lfloor x \rfloor \log x - (x-1) + \int_1^x \frac{\operatorname{frac} t}{t}dt \\
\sum_{n \le x} \log n &= \lfloor x \rfloor \log x - (x-1) + O(\log x)
\end{aligned}$$

□

2. *For each function below determine the growth rate or state that it is 0 on average. In the former case attempt to do the same with the second order term.*

1. $f_1(t) = \Re\left(\zeta\left(\frac{1}{2} + it\right)\right)$
2. $f_2(t) = \Im\left(\zeta\left(\frac{1}{2} + it\right)\right)$
3. $f_3(t) = Z(t)$, *the Hardy Z-function.*

Proof. □

3. *Plot the Hardy Z-function on several intervals. Choose an interval, find all the zeros of the zeta function in that interval and make a polynomial with those same zeros. Do the same with* $\cos(Kt)$ *and determine the scale factor to make the functions more similar.*

Proof. □