REARRANGING SERIES

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Hypothesis. *Let X be an abelian topological group and $\sigma \in S_{\mathbb{N}}$. Let $a = \{a_i\}_{i<\omega} \subset X$ be a sequence. Then the series corresponding to a converges and*

$$\sum_{i<\omega} a_i = \sum_{i<\omega} a_{\sigma(i)}$$

if and only if σ has some easily stated property.

Remark. *Any permutation σ such that $|i - \sigma(i)|$ is bounded preserves every series.*

Remark. *The permutation*

$$\sigma = \prod_{i<\omega} (b_{2i} b_{2i-1})$$

preserves every series where $\{b_i\}_{i<\omega}$ is an increasing sequence such that $b_{2n} - b_{2n-1} > 2$ for infinitely many n.

Proof. For any N such that $b_{2n-1} < N < b_{2n}$ we have that

$$\begin{aligned}
\sum_{i=0}^{N} a_i - \sum_{i=0}^{N} a_{\sigma(i)} &= \sum_{i \in \operatorname{Dom} \sigma \cap N} a_i - \sum_{i \in \operatorname{Dom} \sigma \cap N} a_{\sigma(i)} \\
&= \sum_{i \in \operatorname{Dom} \sigma \cap N} a_i - a_{\sigma(i)} \\
\sum_{i=0}^{N} a_i - \sum_{i=0}^{N} a_{\sigma(i)} &= \sum_{i=0}^{N} a_{b_{2i}} - a_{b_{2i-1}} = 0.
\end{aligned}$$

Therefore every partial sum is the same and so the series is the same. □

Remark. *The previous permutation can be chosen such that $|i-\sigma(i)|$ is not bounded.*

Remark. *There exists a permutation which is the product of transpositions, but does not preserve every series.*

Proof. We build the permutation inductively, that is for each n we will construct τ_n, either a transposition or the identity, such that

$$\sigma = \prod_{i=1}^{\infty} \tau_n \quad \text{and} \quad \sum_{n=1}^{\infty} a_{\sigma(n)} = 0.$$

Let $a_n = (-1)^{n+1}/n$. Define $\tau_0 = e$. At the Nth stage assume that we have defined τ_n for $n \leq N$. Define

$$\sigma_n = \prod_{i=1}^{N} \tau_i \quad \text{and} \quad c_n = \sum_{i=1}^{n} b_n$$

Finally let f_N be the last index such that $\sigma_N \cap \mathbb{N} \setminus f_N$ is the identity. If $N < f_n$ then define $\tau_n = e$ for $N \leq n \leq f_n$ and move to the $f_n + 1$ stage.

Then compute $c_N = \sum_{n=1}^{N} a_{\sigma_N(n)}$. If $c_N > 0$ then let τ_3
while $\sum_{n=1}^{N} a_n > 0$ □

Remark. *There exists a permutation which has unbounded cycle length and preserves all series. In particular*

$$\prod_{i=1}^{\infty}\left(\binom{i}{2}, \binom{i}{2}+1, \ldots, \binom{i+1}{2}-1\right)$$

****.* Given a sequence $\{a_n\}$, and a triangular number t

$$\sum_{i=1}^{t} a_n = \sum_{i=1}^{t} a_{\sigma(n)}.$$

Therefore if the permuted series converges, it converges to the same value as the original series. Let $\epsilon > 0$, $A_n = \sum_{i=1}^{n} a_n$ and $A'_n = \sum_{i=1}^{n} a_{\sigma(n)}$. By the Cauchy criterion there is an N such that $|A_n - A_m| < \epsilon$ for all $n, m > N$. Then N is between two triangular numbers, t and N_0. Then N_0 works for the Cauchy property.

[This is not a proof since we could have easily done something similar with cycles of length 2^n which is demonstrably not Cauchy. □

Remark. *There exists a permutation with an infinite cycle that preserves all series, namely*

$$\sigma = (\cdots 53124 \cdots)$$

We are also interested in sets $B \subset \mathbb{N}$, identified with their characteristic function $B \in 2^{\omega}$, such that

$$F(B) = \sum_{n=0}^{\infty} (-1)^{B(n)} n$$

We use the following notation:

$$\eta_B(x) = |B \cap [0, x]| = \sum_{i=0}^{\lfloor x \rfloor} B(n)$$

$$\delta(B) = \lim_{x \to \infty} \frac{\eta_B(x)}{x}$$

Remark. *It is necessary that $\delta(B) = \frac{1}{2}$.*

****.* WLOG $\delta(B) < \frac{1}{2}$. WLOG $B(N) = 0$. Then for any sufficiently large N we have that for every $M > 0$ $|B_M| < M$ where $B_M = B \cap [N, N + 2M]$. Then for every $1 \in B_M$ there is a preceeding 0. Then we may underestimate the sum by making all these 1s equal in magnitude to the location of their preceeding 0. Then in the interval $[N, N + 2M]$ these values cancel out but there are a positive number of unmatched 0s. This series will be bounded below by $\sum_{n=1}^{\infty} \frac{1}{nM}$ it will diverge. □

Remark. *It is not sufficent that* $\delta(B) = \frac{1}{2}$.

Proof. Consider the set $B = \{b \in \mathbb{N} | b \text{is even or a prime}\}$. Then $\delta(B) = 1/2$ but $F(B)$ diverges. $\square$