RESULTS ON RESTRICTED PARTITIONS

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Notation

The phrase *almost all* means measure zero, therefore any cofinite set in an infinite set comprises almost all the set.

$$p_{A,M}(n) = \left\{ n = \sum_{a \in A} m_a a \middle| m_a \in M \cup 0 \text{ and } m_a = 0 \text{ for almost all } a. \right\}$$

Problem. *How does $p_{A,M}(n)$ behave, given $A, M \subset \mathbb{N}$?*

Problem. *How does $p_{A,M}(n)$ grow asymptotically?*

Problem. *What growth rates are possible under the assumption that $p_{A,M}(n) \geq 1$ for all sufficiently large integers?*

- In [ESG09] Euler showed that if $A = \{2^i\}_{i<\omega}$ and $M = 1$ then $p_{A,M}(n) = 1$ for all n.
- In [Ste58] Stern showed that if $A = \{2^i\}_{i<\omega}$ and $M = 2$ then $p_{A,M}(n) = s(n)$, the Stern sequence, defined recursively by $s(0) = 1, s(2n+1) = s(n), s(2n) = s(n) + s(n-1)$.
- In [ADRW11] Reznick et. al. showed that if $A = \{b^i\}_{i<\omega}$ and M is finite then $p_{A,M}(n)$ has a recursive definition similar to the Stern sequence.
- In [Pro00] Protasov showed that if $A = \{b^i\}_{i<\omega}$ and $M = d$ then $p_{A,M}(n)$ has polynomial growth and further explored the case of $b = 2$ in [Pro04].
- In [CW12] Canfield and Wilf showed that if A is infinite and $M = \mathbb{N}$ then $p_{A,M}(n)$ is superpolynomial, that is, $p_{A,M}(n) = O(n^k)$ is false for all k.
- In [LN11] Ljujic and Nathanson showed that if A and M are infinite then $p_{A,M}(n)$ can be 1 for arbitrarily large n.
- In [Alo12] Alon showed that there exist infinite A and M such that $p_{A,M}(n) = 1$ for all n.

Remark. *If $A = \mathbb{N}$ and $M = 1$ then $p_{A,M}(n)$ is the number of partitions of n into distinct parts*

	$M=1$	$M=2$	M infinite	$M=\mathbb{N}$
A				
A infinite			[LN11], [Alo12]	superpolynomial [CW12]
$A=\{2^i\}_{i<\omega}$	1 [ESG09]	$s(n)$ [Ste58]		$b_\infty(n)$ [Knu66b]
$A=\mathbb{N}$				$p(n)$ [HR18]

Theorem. *Let $A=\{2^i\}_{i<\omega}$ and $M=\mathbb{N}$ and define $b_\infty(n)=p_{A,M}(n)$. Then for all k and sufficiently large n,*

$$n^k < b_\infty(n).$$

Proof. Let N be such that $\left(2+\frac{1}{N}\right)^{k+1} \le 2^{k+1}+1$, and let

$$a = \min\left\{\frac{b_\infty(2n)}{n^{k+1}}\middle| N \le n \le 2N\right\}.$$

Then by induction $b_\infty(2n) \ge an^{k+1}$ for all $n \ge N$ since this is true for $N \le n \le 2N$ and if $n > 2N$

$$\begin{aligned} b_\infty(2n) &= b_\infty(2(n-1)) + b_\infty(n) \\ &\ge a(2(n-1))^{k+1} + an^{k+1} \\ &\ge a\left((2(n-1))^{k+1} + (n-1)^{k+1}\right) = a\left((2(n-1))^{k+1} + (n-1)^{k+1}\right) \\ &\ge a\left(1+\frac{1}{2N}\right)^{k+1}(n-1)^{k+1} \\ &\ge a\left(1+\frac{1}{n-1}\right)^{k+1}(n-1)^{k+1} = an^{k+1} \\ b_\infty(2n) &\ge an^{k+1}. \end{aligned}$$

If we choose $N_k \ge \frac{1}{a}$ and $N_k \ge N$ then the proof is complete. □

Some references

[AS64] [Lat06] [Rez90] [Röd70]

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