PROPERTIES OF STERN-LIKE SEQUENCES

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Let c be an ordinal. If c is finite, associate it to the corresponding integer $c = |c|$. This is an overloaded notation, but context will determine what type of object c is being considered at the time. For example in the two cases $0 \in c$ and $0 < c$ the symbol c is a set and an integer respectively. Let s be a sequence, that is $s = (s_i)_{i \in I}$. Denote the length of s as $|s| = |I|$. Further if $s = (s_i)_{i \in I}$ and $i' \in I$ then $s(i') = s_{i'}$. Denote by $c^{<\omega}$ the set of all finite sequences $s = (s(i))_{i \in |s|}$ where $s(i) \in c$, that is, $c^{<\omega} = \left\{ s \middle| |s| < \omega, s = (s(i))_{i \in |s|} \text{ where } s(i) \in c \right\}$. Let $s, s' \in c^{<\omega}$. Denote concatenation by juxtaposition, that is $s'' = ss' = \{s''(i)\}_{i \in |s|+|s'|}$ where $s''(i) = s'(i)$ if $i < |s'|$ and $s''(i) = s(i - |s'|)$ otherwise. If $s \in c^{<\omega}$ is written adjacent to $i < c$ then i is considered to be a sequence of length 1 and concatenation as defined above applies. Sequences are ordered lexicographically, denoted $\underset{lex}{<}$ with the added definition that if $|s| < |s'|$ then $s \underset{lex}{<} s'$.

Definition (Stern-like sequences)**.** *Let $b \geq 2$ be an integer. Define $S_b(n)$ recursively with*

$$\begin{aligned} S_b(0) = S_b(1) = \cdots = S_b(b-1) &= 1 \\ S_b(bn+i) &= S_b(n) \text{ for } 0 < i < b \\ S_b(bn) &= S_b(n) + S_b(n-1). \end{aligned}$$

Definition (Place Value Partition)**.** *Let c be a positive integer and $s \in c^{<\omega}$. Then s is a place value partition base b of n where*

$$n = pve(s, b) = \sum_{i \in |s|} s(i) b^i.$$

The set of place value partitions of n base b carrying at c is

$$pvp(n, b, c) = \left\{ s \middle| n = pve(s, b), s \in c^{\omega} \right\}.$$

Further define the number of place value partitions of n base b carrying at c as

$$pvr(n, b, c) = |pvp(n, b, c)|$$

and the frequency of occurences of m from n' to n'' as

$$pvrf(m, n', n'', b, c) = \left| \left\{ n \middle| pvr(n, b, c) = m, n' \leq n \leq n'' \right\} \right|$$

Lemma. *The usual b-ary partition of n is lexicographically greatest among $pvp(n, b, c)$ when $c \geq b$.*

Proof. Let s be the usual b-ary partition of n. If $|s| = 1$ then $pvp(n, b, c) = \{s\}$ and the claim is true. Assume that the claim is true for all $|s'| < m$ and that $|s| = m$. Let s' be such that $s \neq s'$ and $n = pve(s', b, c)$. If $|s'| < |s|$ then $s \underset{lex}{<} s'$ and there is nothing to show. If $|s'| = |s|$ then either $s'(|s|) = s(|s|)$ or not. If $|s'| = |s|$ then s and s' share a common prefix, namely y, that is,

$$s = yw_s \quad \text{and} \quad s' = yw_{s'}.$$

But then w_s is a b-ary partition such that $|w_s| < |s|$, and therefore the induction hypothesis applies. Otherwise $|s| = |s'|$ and $s(|s|) \neq s'(|s|)$. If $s'(|s|) < s(|s|)$ then $s' \underset{lex}{<} s$ and there is nothing to show. Finally if $s(|s|) < s'(|s|)$ then $pve(s', b) > n$ since s is the b-ary representation of n. That is, $n \leq s'(|s|)b^{|s|}$. This contradicts our choice of s' hence The usual b-ary partition of n is lexicographically greatest element of $pvp(n, b, c)$ when $c \geq b$. □

Theorem. *For all integers b and n such that $b > 1$ and n nonnegative*

$$pvr(n, b, b + 1) = S_b(n).$$

Proof. For brevity let $A_b(n) = pvr(n, b, b + 1)$. Note that the claim is true for $n < b$ by definition. Assume the induction hypothesis, that is $A_b(m) = S_b(m)$, holds for all $m < n$. Let $r \in b$ such that $r \equiv n \pmod b$. There are two cases, one where $r = 0$ and the other where $r > 0$. Let n' be such that $n = n'b + r$, $a = A_b(n)$ and $a' = A_b(n')$. Enumerate the place value representations of n and n' as $\{s_i\}_{i \in a}$ and $\{s'_i\}_{i \in a'}$ respectively.

Assume first that $r > 0$. Thus $pve(s'_i r, b) = n$ for all $i \in a$ hence $A_b(n') \leq A_b(n)$. Note also that $s''_i = (s_i(j + 1))_{j=0}^{|s|-1} \in pvp(n', b, b + 1)$ since $pve(s''_i, b) = n'$. Further these are distinct because $s_i(0) = r$ for all $i \in a$. Therefore $A_b(n) \leq A_b(n')$, so

$$pvr(n, b, b + 1) = A_b(n) = A_b(n') = S_b(n') = S_b(n'b + r) = S_b(n)$$

when $r > 0$.

If $r = 0$ then for each i either $s_i(0) = 0$ or $s_i(0) = b$. Partition $pvp(n, b, b + 1)$ into

$$\begin{aligned} C_0 &= pvp(n, b, b + 1) \cap \left\{ s \in (b + 1)^{<\omega} \middle| s(0) = 0 \right\} \quad \text{and} \\ C_b &= pvp(n, b, b + 1) \cap \left\{ s \in (b + 1)^{<\omega} \middle| s(0) = b \right\}. \end{aligned}$$

If $s_i(0) = 0$ then $s''_i = (s_i(j+1))_{j=0}^{|s|-1} \in A_b(n')$, so $C_0 \subset pvp(n', b, b+1)$ and s''_i is distinct for each i, so $|C_0| \leq A_b(n')$. Further for $s' \in C_b(n')$ it is the case that $pve(s'0, b) = bn' = n$ therefore $s' \in C_0$ hence $A_b(n') \leq |C_0|$, consequently $|C_0| = A_b(n')$. If $s_i(0) = b$ then $s''_i = (s_i(j + 1))_{j=0}^{|s|-1} \in A_b(n' - 1)$, so similarly $|C_b| = A_b(n' - 1)$. Therefore $A_b(n) = |C_0| + |C_b| = A_b(n') + A_b(n' - 1)$.

□

Lemma.
$$pvrf\left(1, \frac{b^n-1}{b-1}+1, \frac{b^{n+1}-b}{b-1}, b, b+1\right) = (b-1)^n$$

Proof. This can be seen by induction on n. Define $a_n = \frac{b^n-1}{b-1}$. Define $F(i,n) = pvrf(i, a_n, a_{n+1}-1, b, b+1)$. Note that $F(1,0) = 1$. Therefore the induction hypothesis is then that $F(1,n) = (b-1)^n$. Assume that this holds for all $m \leq n$. Then for all i such that $a_n + 1 \leq i \leq a_{n+1}$ and $pvr(i, b, b+1) = S_b(i) = 1$ it is the case that $b \nmid i$. Otherwise $S_b(i)$ would be the sum of two positive values and hence greater than 1. Further, for each such i and $1 \leq j < b$,

$$a_{n+1} + 1 \leq bi + j \leq a_{n+2}$$

and by the recurrence, $pvr(bi+j, b, b+1) = S_b(bi+j) = S_b(i) = 1$. Therefore

$$F(1, n+1) = (b-1)F(1,n) = (b-1)^{n+1}$$

and the induction hypothesis holds for $n+1$. Thus

$$pvrf\left(1, \frac{b^n-1}{b-1}+1, \frac{b^{n+1}-1}{b-1}, b, b+1\right) = (b-1)^n$$

for all n. $\square$