PROPERTIES OF STERN-LIKE SEQUENCES

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Notation

Let c be an ordinal. If c is finite, associate it to the corresponding integer $c = |c|$. This is an overloaded notation, but context will determine what type of object c is being considered at the time. For example in the two cases $0 \in c$ and $0 < c$ the symbol c is a set and an integer respectively. Let s be a sequence, that is $s = (s_i)_{i \in I}$. Denote the length of s as $|s| = |I|$. Further if $s = (s_i)_{i \in I}$ and $i' \in I$ then $s(i') = s_{i'}$. Denote by $c^{<\omega}$ the set of all finite sequences $s = (s(i))_{i \in |s|}$ where $s(i) \in c$, that is, $c^{<\omega} = \left\{ s \middle| |s| < \omega, s = (s(i))_{i \in |s|} \text{ where } s(i) \in c \right\}$. Let $s, s' \in c^{<\omega}$. Denote concatenation by juxtaposition, that is $s'' = ss' = \{s''(i)\}_{i \in |s| + |s'|}$ where $s''(i) = s'(i)$ if $i < |s'|$ and $s''(i) = s(i - |s'|)$ otherwise. If $s \in c^{<\omega}$ is written adjacent to $i < c$ then i is considered to be a sequence of length 1 and concatenation as defined above applies. Sequences are ordered lexicographically greatest index first, denoted $\underset{lex}{<}$ with the added definition that if $|s| < |s'|$ then $s \underset{lex}{<} s'$. In particular $22 \underset{lex}{<} 122 \underset{lex}{<} 200$. The bit shift right operator, >> acts by removing the initial element from a sequence. That is, given $s = (s_i)_{i \in |s|}$ let $\texttt{>>}s = (s(i+1))_{i \in |s|-1}$.

Properties of Stern-like sequences

Definition (Stern-like sequences)**.** *Let $b \geq 2$ be an integer. Define $S_b(n)$ recursively with*

$$S_b(0) = S_b(1) = \cdots = S_b(b-1) = 1$$
$$S_b(bn + i) = S_b(n) \text{ for } 0 < i < b$$
$$S_b(bn) = S_b(n) + S_b(n-1).$$

Definition (Place Value Partition)**.** *Let c be a positive integer and $s \in c^{<\omega}$. Then s is a place value partition base b of n where*

$$n = pve(s, b) = \sum_{i \in |s|} s(i) b^i.$$

The set of place value partitions of n base b carrying at c is

$$pvp(n, b, c) = \left\{ s \middle| n = pve(s, b), s \in c^{\omega}, s(|s|) \neq 0 \right\}.$$

Further define the number of place value partitions of n base b carrying at c as

$$pvr(n, b, c) = |pvp(n, b, c)|$$

and the frequency of occurences of m from n' to n'' as

$$pvrf(m, n', n'', b, c) = \left|\left\{n \middle| pvr(n, b, c) = m, n' \leq n \leq n''\right\}\right|$$

Lemma. *The usual b-ary partition of n is lexicographically greatest among $pvp(n, b, c)$ when $c \geq b$.*

Lemma. *If $r, s, t \in b^{<\omega}, rt = st$ and $t(i) \neq 0$ for all $i \in |t|$ then $S_b(pve(rt, b)) = S_b(pve(st, b))$.*

Corollary. *If the b-ary expansion of n contains no zeroes then $S_b(n) = 1$.*

Theorem. *For all integers b and n such that $b > 1$ and n nonnegative*

$$pvr(n, b, b + 1) = S_b(n).$$

Lemma.

$$pvrf\left(1, \frac{b^n - 1}{b - 1} + 1, \frac{b^{n+1} - b}{b - 1}, b, b + 1\right) = (b - 1)^n$$

Lemma. *Given $n > 0$, if the b-ary expansion of n contains no zeroes then $S_b(n) = 1$.*

The case of $S_3(n)$

Let $a_n = \frac{3^n}{2} + 1$ and $b_n = \frac{3^n}{2}$.

Lemma. *Let $n = 3k + 2$. Then $S_3(n) + S_3(n + 2) = S_3(n + 1)$.*

Proof. This is strictly a derivation based on the recurrence for $S_3(n)$.

$$\begin{aligned} S_3(n + 1) &= S_3(3k + 3) = S_3(3(k + 1)) \\ &= S_3(k) + S_3(k + 1) \\ &= S_3(3k + 2) + S_3(3(k + 1) + 1) \\ S_3(n + 1) &= S_3(n) + S_3(n + 2). \end{aligned}$$

□

Lemma. *Let $m, n > 1$. If there exists i, j such that $m = c_i + j$ and $n = b_i - j$ then $S_3(m) = S_3(n)$.*