PROPERTIES OF STERN-LIKE SEQUENCES

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Notation

Let c be an ordinal. If c is finite, associate it to the corresponding integer $c = |c|$. This is an overloaded notation, but context will determine what type of object c is being considered at the time. For example in the two cases $0 \in c$ and $0 < c$ the symbol c is a set and an integer respectively. Let s be a sequence, that is $s = (s_i)_{i \in I}$. Denote the length of s as $|s| = |I|$. A sequence is considered a $1 \times |s|$ matrix with transposition of s denoted by s^T. Further if $s = (s_i)_{i\in I}$ and $i' \in I$ then $s(i') = s_{i'}$. Denote by $c^{<\omega}$ the set of all finite sequences $s = (s(i))_{i \in |s|}$ where $s(i) \in c$, that is, $c^{<\omega} = \left\{ s \middle| |s| < \omega, s = (s(i))_{i\in|s|} \text{ where } s(i) \in c \right\}$. Let $s, s' \in c^{<\omega}$. Denote concatenation by juxtaposition, that is $s'' = ss' = \{s''(i)\}_{i \in |s|+|s'|}$ where $s''(i) = s'(i)$ if $i < |s'|$ and $s''(i) = s(i - |s'|)$ otherwise. If $s \in c^{<\omega}$ is written adjacent to $i < c$ then i is considered to be a sequence of length 1 and concatenation as defined above applies. Sequences are ordered lexicographically greatest index first, denoted $\underset{lex}{<}$ with the added definition that if $|s| < |s'|$ then $s \underset{lex}{<} s'$. In particular $22 \underset{lex}{<} 122 \underset{lex}{<} 200$. The shift right operator, >> acts by removing the initial element from a sequence. That is, given $s = (s_i)_{i\in|s|}$ let $\gg s = (s(i+1))_{i\in|s|-1}$.

Stern-like sequences

Definition (Stern-like sequences). *Let $b \geq 2$ be an integer. Define $S_b(n)$ recursively with*

$$S_b(0) = S_b(1) = \cdots = S_b(b-1) = 1$$
$$S_b(bn + r) = S_b(n) \text{ for } 0 < r < b$$
$$S_b(bn) = S_b(n) + S_b(n-1).$$

Definition (Place Value Partition). *Let c be a positive integer and $s \in c^{<\omega}$. Then s is a place value partition base b of n where*

$$n = pve(s, b) = \sum_{i\in|s|} s(i)b^i.$$

The set of place value partitions of n base b carrying at c of length at most d is

$$pvp(n, b, c, d) = \left\{ s \middle| n = pve(s, b), s \in c^{\leq d}, s(|s|) \neq 0 \right\}.$$

The set of such partitons of any length is

$$pvp(n,b,c) = \bigcup_{d<\omega} pvp(n,b,c,d).$$

Further define

$$pvr(n,b,c,d) = |pvp(n,b,c,d)| \quad \text{and} \quad pvr(n,b,c) = |pvp(n,b,c)|.$$

Denote the frequency of occurences of m from n' to n'' as

$$pvrf(m,n',n'',b,c) = \left|\left\{n \middle| pvr(n,b,c) = m, n' \le n \le n''\right\}\right|.$$

Theorem. *For all integers b and n such that $b > 1$ and n nonnegative*

$$pvr(n,b,b+1) = S_b(n).$$

Proof. For brevity let $A_b(n) = pvr(n,b,b+1)$. Note that the claim is true for $n < b$ by definition. Assume the induction hypothesis, that is $A_b(m) = S_b(m)$, holds for all $m < n$. Let $r \in b$ such that $r \equiv n \pmod b$. There are two cases, one where $r = 0$ and the other where $r > 0$. Let n' be such that $n = n'b + r$, $a = A_b(n)$ and $a' = A_b(n')$. Enumerate the place value representations of n and n' as $\{s_i\}_{i\in a}$ and $\{s'_i\}_{i\in a'}$ respectively. Given i, let $s''_i = \gg s_i$ that is, s''_i is the sequence resulting from dropping the 0th index from s_i.

Assume first that $r > 0$. Thus $pve(s'_i r, b) = n$ for all $i \in a$ hence $A_b(n') \le A_b(n)$. Note also that $s''_i \in pvp(n',b,b+1)$ since $pve(s''_i, b) = n'$. Further these are distinct because $s_i(0) = r$ for all $i \in a$. Therefore $A_b(n) \le A_b(n')$, so

$$pvr(n,b,b+1) = A_b(n) = A_b(n') = S_b(n') = S_b(n'b + r) = S_b(n)$$

when $r > 0$.

If $r = 0$ then for each i either $s_i(0) = 0$ or $s_i(0) = b$. Partition $pvp(n,b,b+1)$ into

$$C_0 = \left\{s \in pvp(n,b,b+1) \middle| s(0) = 0\right\} \quad \text{and}$$
$$C_b = \left\{s \in pvp(n,b,b+1) \middle| s(0) = b\right\}.$$

If $s_i(0) = 0$ then $pve(s''_i, b) = n'$. Since each i is associated to a distinct s''_i this shows $C_0 \subset pvp(n',b,b+1)$ and $|C_0| \le A_b(n')$. Further for $s' \in C_b(n')$ it is the case that $pve(s'0, b) = bn' = n$ therefore $s' \in C_0$ hence $A_b(n') \le |C_0|$, thus $|C_0| = A_b(n')$. If $s_i(0) = b$ then $pve(s''_i, b) = n' - 1$, so $s''_i \in pvp(n'-1, b, b+1)$. Then, similarly as above, $|C_b| = A_b(n'-1)$. Therefore $A_b(n) = |C_0| + |C_b| = A_b(n') + A_b(n'-1)$. Consequently

$$pvr(n,b,b+1) = S_b(n)$$

for all b and n such that $b > 1$ and $n \ge 0$. ∎

Theorem. *Let*

$$\begin{aligned} d &= \lfloor \log_b(c-1) \rfloor + 1, \\ w &= \frac{(c-1)(b^d-1)}{b-1}, \\ f &= \left\lfloor \frac{w}{b^d} \right\rfloor, \\ n &= mb^d + j \ \text{ and } n > (f+1)b^d. \end{aligned}$$

Then

$$pvr(n,b,c) = \sum_{i \in f+1} pvr(ib^d + j, b, c, d) pvr(m-i, b, c) \tag{$*$}$$

where $j \in b^d$ is an integer.

Proof. This result may be seen by induction. Assume $(*)$ for all $n \leq m$. Note that $n' \neq n''$ implies that $pvp(n', b, c)$ and $pvp(n'', b, c)$ are disjoint. If $s \in pvp(m-i, b, c)$ and $t \in pvp(ib^d + j, b, c, d)$ then $st \in pvp(n, b, c)$. This defines a concatenation map, say g, from the set

$$A = \bigcup_{i \in f+1} pvp(ib^d + j, b, c, d) \times pvp(m-i, b, c)$$

to $pvp(n, b, c)$ which is injective, therefore

$$\sum_{i \in f+1} |pvp(ib^d + j, b, c, d)||pvp(m-i, b, c)| \leq |pvp(mb^d + j, b, c)|.$$

and in other terms,

$$\sum_{i \in f+1} pvr(ib^d + j, b, c, d) pvr(m-i, b, c) \leq pvr(n, b, c).$$

If $u \in pvp(n, b, c)$ then one may factor u as $u = st$ with $|t| = d$. In this case, there exists an i such that $s \in pvp(m-i, b, c)$, further $i < f+1$ because $w < (f+1)b^d$. Then $mb^d + j - (m-i)b^d = ib^d + j$ implies that $t \in pvp(ib^d + j, b, c, d)$. Thus every element of $pvp(n, b, c)$ has an inverse in A, hence g is bijective. Therefore

$$|pvp(n, b, c)| \leq \sum_{i \in f+1} |pvp(ib^d + j, b, c, d)||pvp(m-i, b, c)|.$$

Consequently $pvr(n, b, c) = \sum_{i \in f+1} pvr(ib^d + j, b, c, d) pvr(m-i, b, c)$. □

Example. *Let $b = 2$ and $c = 5$, then $pvr(n, 2, 5)$ is the $5th$ hyperbinary sequence. Further $d = 3, w = pve(444, 2) = 28, f = 3$ and*

$$X_{2,5} = \begin{pmatrix} 1 & 7 & 8 & 4 \\ 1 & 5 & 5 & 2 \\ 2 & 7 & 7 & 2 \\ 2 & 5 & 5 & 1 \\ 4 & 8 & 7 & 1 \\ 3 & 5 & 4 & 0 \\ 5 & 7 & 5 & 0 \\ 4 & 5 & 3 & 0 \end{pmatrix}.$$

Note that $(pvr(15 - i, 2, 5))_{i\in 4} = (9, 12, 8, 12)$ and

$$(pvr(120 + j, 2, 5))_{j\in 8} = (205, 133, 182, 130, 200, 119, 169, 120) = X_{2,5} \cdot (9, 12, 8, 12)$$

See Table 1 for a list of values of $pvr(n, 2, 5)$ for $1 \leq n \leq 128$.

Properties of Stern-like sequences

Lemma. *The usual b-nary partition of n is lexicographically greatest among $pvp(n, b, c)$ when $b \leq c$.*

Proof. Let s be the usual b-nary partition of n. If $|s| = 1$ then $pvp(n, b, c) = \{s\}$ and the claim is true. Assume that the claim is true for all $|s'| < m$ and that $|s| = m$. Let s' be such that $s \neq s'$ and $n = pve(s', b, c)$. If $|s'| < |s|$ then $s \underset{lex}{<} s'$ and there is nothing to show. If $|s'| = |s|$ then either $s'(|s|) = s(|s|)$ or not. If $|s'| = |s|$ then s and s' share a common prefix, namely y, that is,

$$s = yw_s \quad \text{and} \quad s' = yw_{s'}.$$

But then w_s is a b-nary partition such that $|w_s| < |s|$, and therefore the induction hypothesis applies. Otherwise $|s| = |s'|$ and $s(|s|) \neq s'(|s|)$. If $s'(|s|) < s(|s|)$ then $s' \underset{lex}{<} s$ and there is nothing to show. Finally if $s(|s|) < s'(|s|)$ then $pve(s', b) > n$ since s is the b-nary representation of n. That is, $n \leq s'(|s|)b^{|s|}$. This contradicts our choice of s' hence the usual b-nary partition of n is the lexicographically greatest element of $pvp(n, b, c)$ when $c \geq b$. □

Lemma (Common suffix property)**.** *If $r, s, t \in b^{<\omega}, |r| = |s|, r(i) \neq 0$ and $s(i) \neq 0$ for all $i \in |r|$ then $S_b(pve(tr, b)) = S_b(pve(ts, b))$.*

Proof. This can be seen by induction on $|r|$. If $|r| = 0$ then $t = t$ and the claim is true. If $|r| > 0$ then $r(0) \neq 0$ and $s(0) \neq 0$ therefore $S_b(tr) = S_b(\texttt{>>}tr)$ and $S_b(ts) = S_b(\texttt{>>}ts)$. Thus $\texttt{>>}tr = t(\texttt{>>}r), \texttt{>>}ts = t(\texttt{>>}s)$ and $|\texttt{>>}s| = |\texttt{>>}r| = |r|-1 < |r|$ therefore the induction hypothesis applies and $S_b(st) = S_b(\texttt{>>}st) = S_b(\texttt{>>}rt) = S_b(rt)$. □

Corollary. *Given $n > 0$, if the b-nary expansion of n contains no zeroes then $S_b(n) = 1$.*

Proof. For $n = 1$ the assertion holds by definition. If $n > 1$ and its b-nary expansion, say s, contains no zeros then >>s is a b-nary expansion which contains no zeroes and is shorter. Therefore by the induction hypothesis $S_b(n) = S_b(pve(s, b)) = 1$. □

Corollary.

$$pvrf\left(1, \frac{b^n - 1}{b - 1} + 1, \frac{b^{n+1} - 1}{b - 1} - 1, b, b + 1\right) = (b - 1)^n$$

Proof. This can be seen by induction on n. Define $u_n = \frac{b^n - 1}{b-1}$. Define $F(i, n) = pvrf(i, u_n, u_{n+1} - 1, b, b+1)$. Note that $F(1, 0) = 1$. Therefore the induction hypothesis is then that $F(1, n) = (b - 1)^n$. Assume that this holds for all $m \leq n$. Then for all i such that $u_n + 1 \leq i \leq u_{n+1}$ and $pvr(i, b, b + 1) = S_b(i) = 1$ it is the case that $b \nmid i$. Otherwise $S_b(i)$ would be the sum of two positive values and hence greater than 1. Further, for each such i and $1 \leq j < b$,

$$u_{n+1} + 1 \leq bi + j \leq u_{n+2}$$

and by the recurrence, $pvr(bi + j, b, b + 1) = S_b(bi + j) = S_b(i) = 1$. Therefore

$$F(1, n + 1) = (b - 1)F(1, n) = (b - 1)^{n+1}$$

and the induction hypothesis holds for $n + 1$. Thus

$$pvrf\left(1, \frac{b^n - 1}{b - 1} + 1, \frac{b^{n+1} - 1}{b - 1} - 1, b, b + 1\right) = (b - 1)^n$$

for all n. □

Definition. *(Layer and related values) Let $a_{b,n} = \frac{b^n-1}{b-1} + 1, b_{b,n} = \frac{b^{n+1}-1}{b-1}$ and $c_{b,n} = b^n - 1$. Note that $ba_{b,n} = a_{b,n+1} + b - 2$.*

$$\begin{aligned}
ba_{b,n} &= b\left(\frac{b^n - 1}{b - 1} + 1\right) = b\left(1 + \sum_{i \in n} b^i\right) \\
&= b + \sum_{i=1}^{n} b^i = b - 2 + \left(1 + \sum_{i \in n+1} b^i\right) \\
&= b - 2 + \left(\frac{b^{n+1} - 1}{b - 1} + 1\right) \\
ba_{b,n} &= b - 2 + a_{b,n+1}
\end{aligned}$$

Finally define the nth layer of the base b Stern-like sequence as

$$Layer_b(n) = \prod_{a_{b,n}}^{b_{b,n}} \mathrm{str}(S_b(i)).$$

Lemma.

$$Layer_b(n) = Layer_b(n-1)X_n$$

Proof. The equivalent result

$$(**) \qquad i \le c_n \Rightarrow S_b(i + a_{b,n+1}) = S_b(i + a_{b,n})$$

may be seen by induction. In the base case $S_b(1) = S_b(2) = 1$ is true by definiton. Assume that $(**)$ is true for $n = m-1$ and $i = bm + b - 2 \le c_n$. Note that $m < c_{n-1}$.

$$\begin{aligned} i = bm + b - 2 &\le c_m \\ bm + b - 1 &\le c_m + 1 \\ m + \frac{b-1}{b} &\le b^{m-1} \\ m - \frac{1}{b} &\le c_{m-1} \end{aligned}$$

Then since $m \in \mathbb{Z} \Rightarrow m \le c_{n-1}$.

$$\begin{aligned} S_b(i + a_{b,m+1}) &= S_b(bm + b - 2 + a_{b,m+1}) \\ &= S_b(b(m + a_{b,m})) \\ &= S_b(m + a_{b,m}) + S_b(m - 1 + a_{b,m}) \\ &= S_b(m + a_{b,m-1}) + S_b(m - 1 + a_{b,m-1}) \text{ by IH} \\ &= S_b(b(m + a_{b,m-1})) \\ &= S_b(bm + ba_{b,m-1}) \\ &= S_b(bm + b - 2 + a_{b,m}) \\ S_b(i + a_{b,m+1}) &= S_b(i + a_{b,m}). \end{aligned}$$

□

Claim.

$$F_b(x) = \sum_{i<\omega} S_b(i)x^i = \prod_{j<\omega} \sum_{i \in b+1} x^{ib^j}$$

Claim.

$$F_b(x) = F_b(x^b) \sum_{i \in b+1} x^i$$

Proof.

$$\begin{aligned}
F_b(x) &= \prod_{j<\omega} \sum_{i\in b+1} x^{ib^j} \\
&= \left(\prod_{0<j<\omega} \sum_{i\in b+1} x^{ib^j} \right) \left(\sum_{i\in b+1} x^i \right) \\
&= \left(\prod_{j'<\omega} \sum_{i\in b+1} (x^b)^{ib^{j'}} \right) \left(\sum_{i\in b+1} x^i \right) \\
F_b(x) &= F_b(x^b) \sum_{i\in b+1} x^i
\end{aligned}$$

□

THE CASE OF $S_3(n)$

Let $a_n = \frac{3^n-1}{2} + 1, b_n = \frac{3^{n+1}-1}{2}$ and $c_n = 3^n - 1$.

Lemma. *For all $n \geq 0$ let $0 \leq j \leq a_n$, then $S_3(c_n + j) = S_3(b_n - j)$.*

Proof. Proceed by induction on n. For $n = 0$ it is the case that $a_1 = 0, b_1 = 2, c_n = 1$ and $S_3(2) = S_3(1)$ which establishes the base case. Now consider the case when $n > 0$. Let m and m' be such that $c_n \leq m \leq m' \leq b_n$, and $m + m' = b_n + c_n$. This condition is equivalent to the premise of the assertion.

Assume m is not a multiple of 3 and define k, k', r and r' by $m = 3k+r$, $m' = 3k'+r'$ where $\{r, r'\} \subset 3$. Therefore $r + r' = 3$ since $m + m'$ is a multiple of 3. Note that this implies $k + k' = b_{n-1} + c_{n-1}$, hence by the induction hypothesis $S_3(k) = S_3(k')$. By the recurrence, $S_3(m) = S_3(k)$ and $S_3(m') = S_3(k')$, so $S_3(m) = S_3(m')$.

On the other hand if m and m' are multiples of 3 then there is an integer q such that $m = c_n + 3q + 1$ and $m' = b_n - 3q - 1$. Thus

$$\begin{aligned}
S_3(m) &= S_3(c_{n-1} + q + 1) + S_3(c_{n-1} + q) \\
S_3(m') &= S_3(b_{n-1} - q) + S_3(b_{n-1} - q - 1)
\end{aligned}$$

and by the induction hypothesis

$$\begin{aligned}
S_3(b_{n-1} - q) &= S_3(c_{n-1} + q) \\
S_3(b_{n-1} - q - 1) &= S_3(c_{n-1} + q + 1)
\end{aligned}$$

hence $S_3(m) = S_3(m')$. Consequently, for all $n \geq 0$ if $0 \leq j \leq a_n$ then $S_3(c_n + j) = S_3(b_n - j)$. □

Lemma. *Let $n = 3k + 2$. Then $S_3(n) + S_3(n + 2) = S_3(n + 1)$.*

Proof. This is strictly a derivation based on the recurrence for $S_3(n)$.

$$\begin{aligned} S_3(n+1) &= S_3(3k+3) = S_3(3(k+1)) \\ &= S_3(k) + S_3(k+1) \\ &= S_3(3k+2) + S_3(3(k+1)+1) \\ S_3(n+1) &= S_3(n) + S_3(n+2). \end{aligned}$$

□

Claim.

$$(3+1)^n = \sum_{i=a_n}^{b_n} S_3(i)$$

Table 1

n	$pvr(n,2,5)$	n	$pvr(n,2,5)$	n	$pvr(n,2,5)$	n	$pvr(n,2,5)$
1	1	41	37	81	87	120	205
2	2	42	52	82	124	121	133
3	2	43	38	83	92	122	182
4	4	44	60	84	144	123	130
5	3	45	37	85	89	124	200
6	5	46	53	86	127	125	119
7	4	47	38	87	90	126	169
8	8	48	66	88	150	127	120
9	6	49	44	89	98	128	208
10	9	50	63	90	135		
11	7	51	47	91	97		
12	12	52	74	92	150		
13	8	53	46	93	90		
14	12	54	66	94	128		
15	9	55	47	95	91		
16	17	56	79	96	157		
17	12	57	52	97	104		
18	18	58	72	98	148		
19	14	59	52	99	110		
20	23	60	81	100	173		
21	15	61	49	101	107		
22	22	62	70	102	154		
23	16	63	50	103	110		
24	28	64	88	104	184		
25	19	65	59	105	121		
26	27	66	85	106	167		
27	20	67	64	107	120		
28	32	68	102	108	186		
29	20	69	64	109	112		
30	29	70	93	110	159		
31	21	71	67	111	113		
32	38	72	114	112	192		
33	26	73	76	113	126		
34	38	74	106	114	178		
35	29	75	77	115	131		
36	47	76	121	116	203		
37	30	77	74	117	124		
38	44	78	106	118	176		
39	32	79	76	119	124		
40	55	80	131				