THE PETRIDIS PROOF OF THE PLÜNNECKE INEQUALITY

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Conjecture (Erdős-Szemerédi). *For all $\epsilon > 0$ and finite nonempty $A \subset \mathbb{R}$ there exists a $C_\epsilon > 0$ such that*

$$C_\epsilon |A|^{2-\epsilon} \leq \max\{|A+A|, |A \cdot A|\}$$

Theorem. *Let $X, A_1, A_2, \ldots A_n$ be finite nonempty subsets of a commutative group. Then*

$$|X|^{n-1}|A_1 + A_2 + \cdots + A_n| \leq |A_1 + X||A_2 + X| \cdots |A_n + X|.$$

Theorem ([R] Theorem 2.3, $j = 1$). *Let h be an integer, A, B finite sets in a commutative group and $|A + B| = \alpha|A|$. There is a nonempty $X \subset A$ such that*

$$|X + hB| \leq \alpha^h |X|.$$

Remark. *In* [P2011b] *Petridis proves a stronger version of this theorem eliminating the need for the Plünnecke inequalities in this proof.*

Lemma. *Let $A, B_0, \ldots B_{h-1}$ be sets in a commutative group M and define α_i so that $|A + B_i| = \alpha_i|A|$. There is a nonempty $X \subset A$ such that*

$$|X + B_0 + \cdots + B_{h-1}| \leq |X| h^h \left(\prod_{i \in h} \alpha_i \right)$$

Corollary. *Let $A, B_0, \ldots B_{h-1}$ be sets in a commutative group M and define α_i so that $|A + B_i| = \alpha_i|A|$. There is a nonempty $X \subset A$ such that*

$$|X + B_0 + \cdots + B_{h-1}| \leq \alpha_0 \alpha_1 \cdots \alpha_{h-1}|X|.$$

Definition (Layered Graph). *A directed graph $G = G(V, E)$ is a graph with layers $\{V_i\}_{i \in h+1} \subset V$ if $V = \cup_{i \in h+1} V_i$ is a disjoint union and for every edge $(v, v') \in E$ there exists an i such that $0 \leq i < h, v \in V_i$ and $v' \in V_{i+1}$.*

addition graph, $G_+(A, B)$

Definition (Addition Graph). *Let A and B be subsets of a group. Define*

$$V_i = (A + iB) \times \{i\}$$

and

$$E_i = \left\{ \left((v, i), (v + b, i + 1)\right) \middle| v \in A + iB, b \in B \right\}.$$

Let $G = G(V, E)$ be a graph with

$$V = \bigcup_{i \in h+1} V_i \quad \textit{and} \quad E = \bigcup_{i \in h} E_i.$$

Then $G_+(A,B) = G$ is the h-layer addition graph of A and B.

Let G be an addition graph. The *Plünnecke condition* is exactly the structure of this graph that is induced by the commutativity of addition.

Definition (Plünnecke Condition)**.** *Let $G = G(V,E)$ be a layered graph with layers $\{V_i\}_{i\in h}$. Further let j and k be such that $0 < j < h$ and $k \geq 2$. Let vertices $u \in V, v \in V_j$ and $\{w_i\}_{i\in k}$ satisfy $E(u,v)$ and $E(v,w_i)$ for all $i \in k$. If this is the case then there exists k distinct verticies $\{v_i\}_{i\in k} \subset V_j$ such that $E(u,v_i)$ and (v_i,w_i) for all $i \in k$.*

Definition (Path)**.** *Let $G = G(V,E)$ be a graph. A path of length l is a sequence of vertices $(v_i)_{i\in l-1} \in V^l$ such that $E(v_i,v_{i+1})$ for $0 \leq i < l$. Let $P_l(G)$ be the set of paths in a graph of length l, and $P(G)$ be the union over all $l \in \mathbb{N}$. Further, given a path $s = (v_i)_{i\in l-1} \in P_l(G)$ let the ith node in the path s be denoted by $n_i(s) = v_i$.*

Definition (Image, Inverse Image)**.** *Let $G = G(V,E)$ be a graph. The lth image of X is* *image*, $\mathrm{Im}^l(X)$

$$\mathrm{Im}^l(X) = \left\{ v \in V \,\middle|\, \text{there exists } s \in P_l(G) \text{ such that } n_0(s) \in X \text{ and } n_{l-1}(s) = v \right\}.$$

Further, let $\mathrm{Im}^0(X) = X, \mathrm{Im}(X) = \mathrm{Im}^1(X)$. Similarly define the lth inverse image of X as

$$\mathrm{Im}^{-l}(X) = \left\{ v \in V \,\middle|\, \text{there exists } s \in P_l(G) \text{ such that } n_0(s) = v \text{ and } n_{l-1}(s) \in X \right\}.$$

It is now possible to define the magnification ratio.

Definition (Magnification Ratio)**.** *Let G be a finite layered graph with layers $\{V_i\}_{i\in h+1}$. The ith magnification ratio of G is* *magnification ratio*, $D_i(G), D_h, \mu$

$$D_i(G) = \min \left\{ \frac{|\mathrm{Im}^i(Z)|}{|Z|} \,\middle|\, Z \neq \varnothing, Z \subset A \right\}.$$

If G is the primary graph under consideration then let $D_h = D_h(G)$ and $\mu = D_h^{1/h}$.

Definition (Reverse Graph)**.** *Let $G = G(V,E)$ be a directed graph. The reverse graph $G^T = G^T(V,E^T)$ is a graph such that $E^T(y,x)$ if and only if $E(x,y)$. The reverse graph is also referred to as the transpose, as the adjacency matrix of G is the transpose of the adjacency matrix of G^T.*

Definition (Commutative Graph)**.** *A graph G is commutative, that is, a Plünnecke graph, if G satisfies the Plünnecke condition and G^T satisfies the Plünnecke condition.*

Remark. *Therefore if G is commutative then G^T is as well since $(G^T)^T = G$ satisfies the Plünnecke condition.*

And now, the statement of the Plünnecke inequalities.

the Plünnecke inequalities

Theorem ([P2011a] Theorem 1.1)**.** *Let G be a finite commutative graph. $\mu^i \leq D_i(G)$ for all $i \in h+1$.*

incoming degree, outgoing degree, $d_G^-(v), d_G^+(v)$

Definition (Vertex Degrees)**.** *Let $G = G(V, E)$ be a graph. The incoming degree of a vertex $v \in V$ is*

$$d_G^-(v) = \left|\left\{w \in V \middle| E(v, w)\right\}\right|.$$

Similarly the outgoing degree of v is

$$d_G^+(v) = \left|\left\{w \in V \middle| E(w, v)\right\}\right|.$$

Usually the subscript will be dropped when evaluating the degree of a vertex in the primary graph under consideration.

Lemma ([N] Lemma 7.1)**.** *Let $G = G(V, E)$ be a commutative graph with layers $\{V_i\}_{i \in h}$ such that $E(u, v)$. In this case*

$$d^+(u) \geq d^+(v) \quad \text{and} \quad d^-(u) \leq d^-(v).$$

Remark. *Note that if X and Y are sets of vertices in a graph then*

$$\operatorname{Im}(X) \cup \operatorname{Im}(Y) = \operatorname{Im}(X \cup Y).$$

Zone, $Z_n^-(V), Z_n^+(V)$

Definition (Zone of Flow)**.** *Let $G = G(V, E)$ be a graph. For $U \subset V$ define the zone with inflow n of U as*

$$Z_n^-(U) = \left\{u \in U \middle| d^-(u) = n\right\}$$

and similarly define the zone with outflow n of U

$$Z_n^+(U) = \left\{u \in U \middle| d^+(u) = n\right\}.$$

Lemma ([P2011a] Remark)**.** *Let $G = G(V, E)$ be a graph with layers $\{V_i\}_{i \in h}$. Let $i' \in h$ and $k = \sup\left\{d^-(v) \middle| v \in V_{i'}\right\}$. If k is finite and for all $i \leq k$*

$$X_i = Z_i^-(V_{i'}) \quad \text{and} \quad T_i = \operatorname{Im}(X_i) \setminus \left(\bigcup_{i<j\leq k} T_j\right) \quad \text{then}$$

$$\bigcup_{i<j\leq k} \operatorname{Im}(X_j) = \bigcup_{i<j\leq k} T_j. \tag{1}$$

Lemma (PL3.5a)**.** *Let $C \in \mathbb{R}$, H a finite commutative graph with layers $\{U_i\}_{i \in 2}$. If $S \subset U_1 \Rightarrow C|S| \leq |\operatorname{Im}(S)|$ for all $S \subset V$ then*

$$|E(U_0, U_1)| \leq C^{-1}|E(U_1, U_2)|.$$

Corollary (PL3.7)**.** *Let $C \in \mathbb{R}$, H a finite commutative graph with layers $\{U_i\}_{i \in 2}$ such that*

$$S \subset U_1 \Rightarrow C|S| \leq |\operatorname{Im}(S)| \quad \text{and} \quad |E(U_0, U_1)| = C^{-1}|E(U_1, U_2)|.$$

Further let $k = \max\left\{d^-(v)\middle|v \in U_1\right\}$. *If* $1 \le i \le k$, $X_i = Z_i^-(U_1)$ *and* $Y_i = Z_i^-(U_2)$ *then*

$$C|X_i| = |Y_i| \quad \textit{and hence} \quad C|U_1| = |U_2|.$$

Lemma (PL3.5b). *Let* $C_H \in \mathbb{R}$, H *a finite commutative graph with layers* $\{W_i\}_{i\in 2}$ *such that*

$$S \subset W_1 \Rightarrow C_H|S| \le |\operatorname{Im}(S)| \quad \textit{and} \quad S \subset W_1 \Rightarrow C_H^{-1}|S| \le |\operatorname{Im}^{-1}(S)|$$

Then

$$|E(U_0, U_1)| = C^{-1}|E(U_1, U_2)|.$$

Lemma (PL3.5c). *Let* $C \in \mathbb{R}$, H *a finite commutative graph with layers* $\{W_i\}_{i\in 2}$ *Assume* H *is such that for all* $S \subset W_1$

$$C|S| \le |\operatorname{Im}(S)| \quad \textit{and} \quad C^{-1}|S| \le |\operatorname{Im}^{-1}(S)|.$$

Let $k = \max\left\{d^-(v)\middle|v \in W_1\right\}$. *If* $1 \le i \le k$, $X_i = Z_i^-(W_1)$ *and* $Y_i = Z_i^-(W_2)$ *then*

$$C|X_i| = |Y_i| \quad \textit{and hence} \quad C|W_1| = |W_2|.$$

Corollary. *Let* $C_H \in \mathbb{R}$, H *a finite commutative graph with layers* $\{U_i\}_{i\in 2}$. *Assume* H *is such that for all* $S \subset U_1$

$$C_H|S| \le |\operatorname{Im}(S)| \quad \textit{and} \quad C_H^{-1}|S| \le |\operatorname{Im}^{-1}(S)|.$$

Then

$$C_H|U_0| = |U_1|.$$

Lemma ([P2011a] Lemma 3.6). *Let* $C \in \mathbb{R}$ *such that* $C > 1$, H *a finite weighted commutative graph with layers* $\{U_i\}_{i\in 2}$ *and* $w(v) = C^{-i}$ *for all* $v \in U_i$. *If* U_1 *is a separating set of minimum weight then* U_0 *is a separating set of minimum weight.*

Lemma ([P2011a] Lemma 3.3). *Let* $C \in \mathbb{R}$, G *a finite commutative graph with layers* $\{V_i\}_{i\in h}$ *and* $w(v) = C^{-i}$ *for all* $v \in V_i$. *There exists a separating set of minimum weight contained in* $V_0 \cup V_h$.

Corollary ([P2011a] Corollary 3.4). *Let* G *be a finite commutative graph with layers* $\{V_i\}_{i\in h}$ *and* $w(v) = \mu^{-i} = D_h(G)^{-i/h}$ *for all* $v \in V_i$. *Let* S *be a separating set of minimum weight. Then the weight of* S *is* $w(S) = |V_0|$.

Lemma (Direct Product Lemmma). *If* A, B, C, D *are nonempty subsets of a group,* $|A + B| = \alpha|A|$, *and* $|C + D| = \beta|C|$ *then* $|A \times C + B \times D| = \alpha\beta|A \times C|$.

Definition (Layered Graph Product, [R] Definition 4.1). *Let* $G' = G(V', E')$, $G'' = G(V'', E'')$, *be graphs with layers* $\{V_i'\}_{i\in h+1} \subset V'$ *and* $\{V_i''\}_{i\in h+1} \subset V''$ *respectively. The layered product of* G' *and* G'' *is* $G = G(V, E) = G'G''$ *where* $V_i = V_i' \times V_i''$ *form the layers of* G *and for* $(u', u''), (v', v'') \in V \times V''$ *let* $E((u', u''), (v', v''))$ *be true if and only if* $E'(u', v')$ *and* $E(u'', v'')$.

Remark. *The layered graph product is a proper subgraph of the usual product graph.*

Lemma ([R] Lemma 4.2)**.** *The layered graph product of commutative graphs is commutative.*

Lemma (Multiplicativity of Magnification Ratios, [R] Lemma 4.3)**.** *If* $G = G(V, E), G' = G(V', E')$ *and* $G'' = (V'', E'')$ *are graphs with layers* $\{V_i\}_{i \in h+1} \subset V$*,* $\{V'_i\}_{i \in h+1} \subset V'$ *and* $\{V''_i\}_{i \in h+1} \subset V''$ *respectively and* $G = G'G''$ *then* $D_i(G) = D_i(G')D_i(G'')$ *for all* $i \in h$*.*

References

[N] M. B. Nathanson. *Additive Number Theory: Inverse Problems and the Geometry of Sumsets.* Springer, New York, 1996.

[P2011a] G. Petridis. Plünnecke 's inequality
`http://arxiv.org/abs/1101.2532`

[P2011b] G. Petridis. New proofs of Plünnecke -type Estimates for Product Sets in Groups
`http://arxiv.org/abs/1101.3507`

[R] I. Z. Ruzsa. Sumsets and structure. In *Combinatorial Number Theory and Additive Group Theory*. Springer, New York, 2009.
`http://math.cmu.edu/~af1p/Teaching/AdditiveCombinatorics/Additive-Combinatorics.pdf`