AN ESTIMATE ON SUM SETS

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Theorem. *Let $X, A_1, A_2, \dots A_n$ be finite nonempty subsets of a commutative group. Then*
$$|X|^{n-1}|A_1 + A_2 + \cdots + A_n| \leq |A_1 + X||A_2 + X| \cdots |A_n + X|.$$

Definition (Layered Graph)**.** *A directed graph $G = G(V, E)$ is a graph with layers $\{V_i\}_{i \in h+1} \subset V$ if $V = \cup_{i \in h+1} V_i$ is a disjoint union and for every edge $(v, v') \in E$ there exists an i such that $0 \leq i < h, v \in V_i$ and $v' \in V_{i+1}$.*

magnification ratio, $D_i(G), D_h, \mu$

Definition (Magnification Ratio)**.** *Let G be a finite layered graph with layers $\{V_i\}_{i \in h+1}$. The ith magnification ratio of G is*
$$D_i(G) = \min \left\{ \frac{|\operatorname{Im}^i(Z)|}{|Z|} \middle| Z \neq \varnothing, Z \subset A \right\}.$$
If G is the primary graph under consideration then let $D_h = D_h(G)$ and $\mu = D_h^{1/h}$.

the Plünnecke inequalities

Theorem ([P2011a] Theorem 1.1)**.** *Let G be a finite commutative graph and define $\mu = (D_h(G))^{1/h}$. Then $\mu^i \leq D_i(G)$ for all $i \in h+1$.*

Definition (Addition Graph)**.** *Let A and B be finite subsets of a group. Define*
$$V_i = (A + iB) \times \{i\}$$
and
$$E_i = \left\{ ((v, i), (v + b, i + 1)) \middle| v \in A + iB, b \in B \right\}.$$
Let $G = (V, E)$ be a graph with
$$V = \bigcup_{i=0}^{h} V_i \quad \text{and} \quad E = \bigcup_{i=0}^{h-1} E_i.$$
Then $G_{+,h}(A, B) = G$ is the addition graph of A and B with layers $\{V_i\}_{i \in h+1}$.

Lemma (Direct Product Lemmmma)**.** *If A, B, C, D are nonempty subsets of a group, $|A + B| = \alpha|A|$, and $|C + D| = \beta|C|$ then $|A \times C + B \times D| = \alpha\beta|A \times C|$.*

Proof. Note that $(x, y) \in A \times C + B \times D$ if and only if there exists $a \in A, b \in B, c \in C, d \in D$ such that $x = a + b$ and $y = c + d$. Thus
$$|A \times C + B \times D| = |A + B||C + D| = \alpha\beta|A||C|$$
$$|A \times C + B \times D| = \alpha\beta|A \times C|.$$
□

Definition (Layered Graph Product, [R] Definition 4.1)**.** *Let $G' = G(V', E')$, $G'' = G(V'', E'')$, be graphs with layers $\{V_i'\}_{i \in h+1} \subset V'$ and $\{V_i''\}_{i \in h+1} \subset V''$ respectively. The layered product of G' and G'' is $G = G(V, E) = G'G''$ where $V_i = V_i' \times V_i''$ form the layers of G and for $(u', u''), (v', v'') \in V \times V''$ let $E((u', u''), (v', v''))$ be true if and only if $E'(u', v')$ and $E(u'', v'')$.*

Remark. *Note that the layered graph product is a proper subgraph of the usual product graph.*

Lemma ([R] Lemma 4.2)**.** *The layered graph product of commutative graphs is commutative.*

Proof. Using the notation in the definition of the layered graph product, assume G' and G'' are commutative. If $E(u, v)$ and $E(v, w_i)$ for $u, v, w_i \in V$ and $i \in k$ then $\pi_0(u) \in V', \pi_0(v) \in V', \pi_0(w_i) \in V'$ for $i \in k$, $E(\pi_0(u), \pi_0(v))$ and $E(\pi_0(v), \pi_0(w_i))$ for each $i \in k$. By the commutativity of G' there exist k distinct vertices $\{v_i'\}_{i \in k} \subset V'$ such that $E(\pi_0(u), v_i')$ and $E(v_i', \pi_0(w_i))$ for each $i \in k$. Similarly there exist k distinct vertices $\{v_i''\}_{i \in k} \subset V''$ satisfying the same property in the second coordinate. Therefore there exist k distinct vertices $\{v_i = (v_i', v_i'')\}_{i \in k} \subset V$ such that $E(u, v_i)$ and $E(v_i, w_i)$ for all $i \in k$. Consequently G is commutative. $\square$

Lemma (Multiplicativity of Magnification Ratios, [R] Lemma 4.3)**.** *If $G = G(V, E), G' = G(V', E')$ and $G'' = (V'', E'')$ are graphs with layers $\{V_i\}_{i \in h+1} \subset V$, $\{V_i'\}_{i \in h+1} \subset V'$ and $\{V_i''\}_{i \in h+1} \subset V''$ respectively and $G = G'G''$ then $D_i(G) = D_i(G')D_i(G'')$ for all $i \in h$.*

Proof. Note that $D_i(G) \leq D_i(G')D_i(G'')$ since choosing sets $Z' \subset V_0'$ and $Z'' \subset V_0''$ which achieve the minimum magnification in G' and G'' respectively yields $Z = Z' \times Z'' \subset V_0$ demonstrating the upper bound.

To see that $D_i(G')D_i(G'') \leq D_i(G)$, consider the graph $G'' = G_{+,1}(W, \{e\})$ where W is a finite set and e is the identity of the group under consideration, and note that $\mu'' = D_1(G'') = 1$. Now given $X \subset V_0 = V_0' \times W$ define

$$X_w = \left\{(a, b) \in V_0 \middle| b = w \text{ for some } w \in W\right\}$$

and $\mu' = D_1(G')$. These sets form a partition of X, in particular,

$$X = \bigcup_{w \in W} X_w$$

and consequently

$$|\operatorname{Im}(X)| = \sum_{w \in W} |\operatorname{Im}(X_w)| \geq \sum_{w \in W} \mu' |X_w| = \mu' |X|$$

which gives the upper bound in the case $\mu'' = 1$.

For the general case, construct a graph $H = G(V_H, E_H)$ from the layers

$$U_0 = V_0' \times V_0'' = V_0, U_1 = V_j' \times V_0'', U_2 = V_j' \times V_j'' = V_j.$$

Let $u_0 = (x', x'') \in U_0, u_1 = (y', x'') \in U_1$ and $E_H(u_0, u_1)$ if and only if there is a path in G' from x' to y'. Similarly let $u_1 = (y', x'') \in U_1, u_2 = (y', y'')$ and $E_H(u_1, u_2)$ if and only if there is a path in G'' from x'' to y''. Note there is a path from u_0 to u_2 in H if and only if there is a path from u_0 to u_2 in G, hence

$$D_2(H) = D_j(G).$$

Then the induced subgraphs H_1, H_2 on $U_0 \cup U_1$ and $U_1 \cup U_2$ respectively are examples of the case treated before, so

$$D_1(H_1) \geq D_j(G') \quad \text{and} \quad D_1(H_2) \geq D_j(G'')$$

and finally

$$D_j(G) = D_2(H) \geq D_1(H_1) D_2(H_2) \geq D_j(G') D_j(G'').$$

Consequently,

$$D_j(G) = D_j(G') D_j(G'').$$

□

Corollary ([R] Theorem 2.3, $j = 1$)**.** *Let h be an integer, A, B finite sets in a commutative group and $|A + B| = \alpha |A|$. There is a nonempty $X \subset A$ such that*

$$|X + hB| \leq \alpha^h |X|.$$

Proof. Let $G = G_{+,h}(A, B)$. By the definition of magnification ratio

$$D_1(G) = \min \left\{ \frac{|\operatorname{Im}(Z)|}{|Z|} \middle| Z \neq \varnothing, Z \subset A \right\} \leq \frac{|\operatorname{Im}(A)|}{|A|} = \frac{|A + B|}{|A|} = \alpha.$$

Then by the Plünnecke inequality

$$\mu = (D_h(G))^{\frac{1}{h}} \leq D_1(G).$$

But then

$$\mu^h = \min \left\{ \frac{|\operatorname{Im}^h(X)|}{|X|} \middle| X \neq \varnothing, X \subset A \right\} \leq \alpha^h,$$

therefore there exists a nonempty $X \subset A$ such that

$$|X + hB| \leq |X| \alpha^h.$$

□

Remark. *In* [P2011b] *Petridis proves a stronger version of this theorem eliminating the need for the Plünnecke inequalities in this proof.*

Lemma. *Let $A, B_0, \dots B_{h-1}$ be sets in a commutative group M and define α_i so that $|A + B_i| = \alpha_i |A|$. There is a nonempty $X \subset A$ such that*

$$|X + B_0 + \cdots + B_{h-1}| \le h^h \left(\prod_{i \in h} \alpha_i \right) |X|$$

Proof. Since the α_i are rational let n be their least common denominator and define $n_i = n/\alpha_i$. Let $N = M \times \mathbb{Z}^h$. Consider elements in this group $(h+1)$-tuples and let π_i be the value of ith entry. Then define $e_i \in N$ so that $\pi_j(e_i) = \delta_{i,j}$ where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. Let $C = A + B_0 + \cdots + B_{h-1}$ and $T_i = \{(j+1)e_{i+1}\}_{j \in n_i}$ so that $|T_i| = n_i$. Then note that each sum of the form

$$c + t_0 + \cdots + t_{h-1} \quad \text{where } c \in C, t_i \in T_i$$

is distinct for a given assignment of values $(c, t_0, \dots, t_{h-1}) \in C \times \prod_{i \in h} T_i$. Now define $B = \cup_{i \in h} (B_i + T_i)$ and note that $A + B_i + T_i \cap A + B_j + T_j = \varnothing$. Then

$$|A + B| = \sum_{i \in h} |A + B_i + T_i| \le \sum_{i \in h} |A + B_i||T_i| = |A| \sum_{i \in h} n_i \alpha_i = hn|A|.$$

Therefore from the previous corollary there exists a nonempty $X \subset A$ such that

$$|X + hB| \le h^h n^h |X|$$

Further $X + B_0 + \cdots + B_{h-1} + T_0 + \cdots T_{h-1} \subset |X + hB|$ and hence

$$|X + B_0 + \cdots + B_{h-1}| \prod_{i \in h} n_i \le |X + hB|$$

which results in the inequality

$$|X + B_0 + \cdots + B_{h-1}| \le |X| h^h n^h \left(\prod_{i \in h} n_i \right)^{-1}.$$

But

$$h^h n^h = h^h \left(\prod_{i \in h} \alpha_i \right) \left(\prod_{i \in h} n_i \right)$$

therefore the above inequality becomes

$$|X + B_0 + \cdots + B_{h-1}| \le |X| h^h \left(\prod_{i \in h} \alpha_i \right) \left(\prod_{i \in h} n_i \right) \left(\prod_{i \in h} n_i \right)^{-1} = |X| h^h \left(\prod_{i \in h} \alpha_i \right)$$

consequently

$$|X + B_0 + \cdots + B_{h-1}| \le |X| h^h \left(\prod_{i \in h} \alpha_i \right).$$

□

Corollary. *Let $A, B_0, \ldots B_{h-1}$ be sets in a commutative group M and define α_i so that $|A + B_i| = \alpha_i |A|$. There is a nonempty $X \subset A$ such that*

$$|X + B_0 + \cdots + B_{h-1}| \leq \left(\prod_{i \in h} \alpha_i \right) |X|.$$

Proof. Let k be a positive integer. By the direct product lemma if $|A + B_i| = \alpha_i |A|$ then $|A^k + B_i^k| = \alpha_i^k |A^k|$. Now define

$$\begin{array}{rclcrcl} C & = & A + B_0 + \cdots + B_{h-1}, & & C' & = & A^k + B_0^k + \cdots B_{h-1}^k, \\ G & = & G_{+,1}(A, C), & & G' & = & G_{+,1}(A^k, C'), \\ \mu & = & D_1(G) & & \mu' & = & D_1(G') \end{array}$$

Then by the previous lemma

$$\mu \leq h^h \prod_{i \in h} \alpha_i \quad \text{and} \quad \mu' \leq h^h \prod_{i \in h} \alpha_i.$$

Now note that G' is isomorphic to G^k, the kth layered product of G. Given an edge (u, v) of G' by definition there exists $u \in A^k, v \in A^k + C'$ and an element $w \in C'$ such that $u + w = v$. Further w is unique because group inverses are unique. This edge exists if and only if $\pi_i(u) \in A$, $\pi_i(w) \in C$ and $\pi_i(v) \in A + C$ for all $i \in k$. Therefore $(\pi_i(u), \pi_i(v))$ is an edge in G for all $i \in k$, and in G^k there is exactly one edge corresponding to (u, w). This shows G' and G^k are isomorphic since the edge given was arbitrary. Thus by the multiplicativity of magnification ratios $\mu' = \mu^k$ and

$$\mu \leq h^{\frac{h}{k}} \prod_{i \in h} \alpha_i.$$

Since k is arbitrary $\mu \leq \prod_{i \in h} \alpha_i$ must hold, hence there exists a nonempty $X \subset A$ such that

$$|X + B_0 + \cdots + B_{h-1}| \leq \left(\prod_{i \in h} \alpha_i \right) |X|.$$

□

Theorem. *Let $X, A_1, A_2, \dots A_n$ be finite nonempty subsets of a commutative group. Then*

$$|X|^{n-1}|A_1 + A_2 + \cdots + A_n| \leq |A_1 + X||A_2 + X| \cdots |A_n + X|.$$

Proof. By the previous corollary there exists an $X' \subset X$ such that

$$|X' + A_1 + \cdots A_n| \leq |X'| \prod_{i=1}^{n} \frac{|X + A_i|}{|X|} = |X'||X|^{-n} \prod_{i=1}^{n} |X + A_i|$$

But note that $|X'| \leq |X|$ and $|A_1 + \cdots A_n| \leq |X' + A_1 + \cdots A_n|$ so

$$|A_1 + \cdots A_n| \leq |X' + A_1 + \cdots A_n| \leq |X'||X|^{-n} \prod_{i=1}^{n} |X + A_i| \leq |X|^{1-n} \prod_{i=1}^{n} |X + A_i|.$$

Therefore

$$|X|^{n-1}|A_1 + A_2 + \cdots + A_n| \leq |A_1 + X||A_2 + X| \cdots |A_n + X|.$$

□

References

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