AN INEQUALITY DUE TO RUZSA

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Theorem. *Let G be a group and X, A_1, A_2 be finite subsets of G.*

$$|X||A_1 + A_2| \leq |A_1 - X||X + A_2|$$

Proof. Let $A = A_1 + A_2$. Pick $r : A \to A_1 \times A_2$ such that $r(a) = (a_{1,a}, a_{2,a})$ and $a = a_{1,a} + a_{2,a}$. Define a map

$$f : X \times A \to (A_1 - X) \times (X + A_2)$$
$$(x, a) = (x, a_{1,a} + a_{2,a}) \mapsto (a_{1,a} - x, x + a_{2,a}).$$

This map is injective. Let $(x, a), (x', a') \in X \times A$ and $f((x, a)) = f((x', a'))$. Then

$$f((x, a)) = (a_{1,a} - x, x + a_{2,a}) = (a'_{1,a'} - x', x' + a'_{2,a'}) = f((x', a')).$$

Thus

$$a_{1,a} - x = a'_{1,a'} - x' \tag{1}$$
$$x + a_{2,a} = x' + a'_{2,a'}. \tag{2}$$

so (1) + (2) shows $a = a_{1,a} + a_{2,a} = a'_{1,a'} + a'_{2,a'} = a'$. Since r is injective, $r(a) = (a_{1,a}, a_{2,a}) = (a'_{1,a'}, a'_{2,a'}) = r(a')$. In particular $a_{2,a} = a'_{2,a'}$, so (2) implies $x = x'$. Therefore $(x, a) = (x', a')$. Hence f is injective and

$$|X||A_1 + A_2| \leq |A_1 - X||X + A_2|.$$

□