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Asymptotic behaviour of the partition function

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Abstract. Given a pair of positive integers m and d such that $2 \leqslant m \leqslant d$, for integer $n \geqslant 0$ the quantity $b_{m,d}(n)$, called the partition function is considered; this by definition is equal to the cardinality of the set

$$\Bigl\{(a_0, a_1, \dots) : n = \sum_k a_k m^k,\ a_k \in \{0, \dots, d-1\},\ k \geqslant 0\Bigr\}.$$

The properties of $b_{m,d}(n)$ and its asymptotic behaviour as $n \to \infty$ are studied. A geometric approach to this problem is put forward. It is shown that

$$C_1 n^{\lambda_1} \leqslant b_{m,d}(n) \leqslant C_2 n^{\lambda_2}$$

for sufficiently large n, where C_1 and C_2 are positive constants depending on m and d, and $\lambda_1 = \varliminf_{n\to\infty} \dfrac{\log b(n)}{\log n}$ and $\lambda_2 = \varlimsup_{n\to\infty} \dfrac{\log b(n)}{\log n}$ are characteristics of the exponential growth of the partition function. For some pair (m,d) the exponents λ_1 and λ_2 are calculated as the logarithms of certain algebraic numbers; for other pairs the problem is reduced to finding the joint spectral radius of a suitable collection of finite-dimensional linear operators. Estimates of the growth exponents and the constants C_1 and C_2 are obtained.

Bibliography: 17 titles.

§ 1. Introduction

For a pair of positive integers m and d such that $2 \leqslant m \leqslant d$ we consider the quantity $b_{m,d}(n)$ equal to the number of possible partitionings of a fixed integer $n \geqslant 0$ into a sum of powers of m with 'digits' from the set $0, \dots, d-1$:

$$n = a_0 + a_1 m + \dots + a_l m^l,$$

where $l \in \mathbb{N} \cup \{0\}$ and $a_i \in \{0, \dots, d-1\}$ for $i = 0, \dots, l$, $a_l \neq 0$. The function $b_{m,d}(n)$ is called the *partition function* of order d with base m. The partition function of order ∞ is defined by the equality $b_{m,\infty}(n) = \lim_{d\to\infty} b_{m,d}(n)$.

Partition functions are well known in mathematics. Euler considered the binary partition function of infinite order $b_{2,\infty}(n)$ involved in the expansion of the function

$$F(x) = \prod_{j=0}^{\infty} (1 - x^{2^j})^{-1} = \sum_{n=0}^{\infty} b_{2,\infty}(n) x^n.$$

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Arithmetical and analytic properties of partition functions have been studied by many authors such as Tanturri, Mahler, Knuth, de Bruijn, Churchhouse, Reznick (see the bibliography). In particular, explicit formulae are known for binary partition functions with small values of the order d:

(1) $b_{2,2}(n) \equiv 1$ (Euler [1]);
(2) $b_{2,3}(n) = s(n+1)$ (Reznick [2]), where $s(n+1)$ is the so-called Stern sequence which is defined recursively as follows: $s(0) = 0$, $s(1) = 1$, $s(2x) = s(x)$, $s(2x+1) = s(x) + s(x+1)$ (see [3]);
(3) $b_{2,4}(n) = \lfloor n/2 \rfloor + 1$ (Klosinsky, Alexanderson, Hillman [4]), where $\lfloor x \rfloor$ is the largest integer not exceeding x (and $\lceil x \rceil$ is the smallest integer not smaller than x).

Arguably, this list exhausts all 'well-behaved' partition functions. For other values of d there can hardly exist formulae of this simplicity for the calculation of $b_{2,d}(n)$. In these circumstances the most interesting problem is to study the asymptotic behaviour of $b_{2,d}(n)$ as $n \to \infty$. The first result in this direction is due to Mahler [5], who showed that

$$\log_2 b_{2,\infty}(n) \sim \frac{\log_2^2 n}{\log_2 4} \quad \text{as } n \to \infty.$$

This result has been repeatedly improved upon; see, for instance, [6], [7].

Reznick [2] considers the case of finite order d and shows that for $d = 2^r$, $r \geqslant 1$, we have the following asymptotics:

$$b_{2,2^r}(n) \sim c n^{r-1} \quad \text{as } n \to \infty \tag{1}$$

with constant c independent of r. If d is not a power of two, then the asymptotic formula becomes more complicated. For even d we have

$$C_1 n^{\log_2 k} \leqslant b_{2,2k}(n) \leqslant C_2 n^{\log_2 k}, \qquad n \in \mathbb{N}, \tag{2}$$

where the constants C_1 and C_2, $0 < C_1 \leqslant C_2$, depend on k. The picture for odd d is a different one. Reznick considers the following limits, which one could appropriately call the lower and upper growth exponents:

$$\lambda_1 = \varliminf_{n \to \infty} \frac{\log b(n)}{\log n}, \qquad \lambda_2 = \varlimsup_{n \to \infty} \frac{\log b(n)}{\log n}. \tag{3}$$

If $b(n) = b_{2,2k}(n)$, then it follows from (2) that $\lambda_1 = \lambda_2 = \log_2 k$. However, it turns out that for d odd (that is, for $b(n) = b_{2,2k+1}(n)$) the exponents λ_1 and λ_2 are not the same in general. For $d = 3$ these exponents can be explicitly calculated:

$$\lambda_1 = 0, \qquad \lambda_2 = \log_2 \frac{\sqrt{5}+1}{2}. \tag{4}$$

For $d = 2k + 1$, $k \geqslant 2$, it can be shown that λ_1 and λ_2 are positive and finite, but they have not been explicitly calculated for any $k \geqslant 2$, although there exist some estimates [2]. Reznick also could not answer the question on a possible generalization of inequality (2) to odd orders $d \geqslant 5$. In other words, is it true that for $b(n) = b_{2,d}(n)$ the limits

$$\alpha = \varliminf_{n\to\infty} b(n)n^{-\lambda_1}, \qquad \beta = \varlimsup_{n\to\infty} b(n)n^{-\lambda_2} \tag{5}$$

are positive and finite for each $d \geqslant 3$?

In the present paper we consider the general case, that is, partition functions with general base $m \geqslant 2$. For an arbitrary pair (m,d) $(2 \leqslant m \leqslant d-1)$ we prove that the growth exponents λ_1 and λ_2 defined by equalities (3) with $b(n) = b_{m,d}(n)$ are finite. Moreover, if $m+1 \leqslant d \leqslant 2m-1$, then $\lambda_1 = 0$, otherwise $\lambda_1 > 0$. We reduce the problem of the calculation of λ_1 and λ_2 to finding the joint spectral radius of an appropriate family of finite-dimensional linear operators. For some pairs (m,d) we calculate the growth exponents explicitly, while for others we find estimates. In particular, for the binary partition function $(m = 2)$ we find λ_1 and λ_2 for the orders $d = 5, 7, 9, 11$, and 13, and we also formulate a conjecture generalizing this result to all odd orders d.

We prove that the limits α and β defined by (5) for $b(n) = b_{m,d}(n)$ are positive and finite for each pair (m,d). This means, in particular, that we answer Reznick's question in the affirmative: inequality (2) does indeed hold for each pair (m,d). For all m and d such that $2 \leqslant m \leqslant d-1$ there exist positive constants C_1 and C_2 such that

$$C_1 n^{\lambda_1} \leqslant b_{m,d}(n) \leqslant C_2 n^{\lambda_2}, \qquad n \in \mathbb{N}. \tag{6}$$

For each pair (m,d) we obtain estimates of the quantities λ_1, λ_2, α, and β.

In the present paper we put forward a geometric approach to the analysis of partition functions. The central idea is that, in place of the function $b(n)$, we study the vector-valued function

$$v(n) = \bigl(b(n), \dots, b(n-s+1)\bigr)^T \in \mathbb{R}^s,$$

where the dimension s is defined separately for each pair (m,d). The vector $v(n)$ here can be obtained from another vector $v(0)$ by the application of an appropriate sequence of linear operators. Next we study the asymptotic behaviour of the quantity $\|v(n)\|$ as $n \to \infty$ making use of the joint spectral radius and the lower spectral radius of these operators.

The paper falls into several sections. In § 2 we recall the definitions and the basic properties of the joint spectral radius and the lower spectral radius. In §§ 3 and 4 we study some special properties of operators with invariant cone. Next, in § 5, we prove our main result, Theorem 1 on the asymptotic behaviour of the partition function. Finally, in § 6 we develop a method for the calculation of the growth exponents λ_1 and λ_2 and set forth the results of our calculations for $m = 2$ and some odd orders d.

§ 2. Joint spectral radius

Let $A = \{A_0, \dots, A_{m-1}\}$, where $m \in \mathbb{N}$, be a finite collection of linear operators in the Euclidean space $\mathbb{R}^s$, $s \in \mathbb{N}$. For a fixed positive integer ℓ let $\max_\ell A$

and $\min_\ell A$ be the following quantities:

$$\max_\ell A = \max_{\substack{d_j\in\{0,\dots,m-1\}\\ j=1,\dots,\ell}} \|A_{d_1}\cdots A_{d_\ell}\|,$$
$$\min_\ell A = \min_{\substack{d_j\in\{0,\dots,m-1\}\\ j=1,\dots,\ell}} \|A_{d_1}\cdots A_{d_\ell}\|. \tag{7}$$

Definition 1. The limits

$$\hat\rho(A) = \lim_{\ell\to\infty} (\max_\ell A)^{1/\ell},$$
$$\check\rho(A) = \lim_{\ell\to\infty} (\min_\ell A)^{1/\ell}$$

are called the *joint spectral radius* and the *lower spectral radius* of the collection $A = \{A_0,\dots,A_{m-1}\}$, respectively.

The concept of joint spectral radius appeared for the first time in [8], where it was used in a problem in the theory of normed algebras. After that joint spectral radii found many applications in wavelet theory, functional equations, approximation theory, fractals (see the vast bibliography on this subject in [9] and [10]). The concept of lower spectral radius was introduced in [11]. We shall use only the most basic properties of these characteristics.

(1) If $A_0 = \dots = A_{m-1}$, then $\check\rho(A) = \hat\rho(A) = \rho(A_0)$, where $\rho(A_0)$ is the (usual) spectral radius of A_0, that is, the largest absolute value of its eigenvalues.

(2) For an arbitrary collection of operators $\{A_0,\dots,A_{m-1}\}$ we have

$$\hat\rho(A) = \lim_{\ell\to\infty} \max_{(d_1,\dots,d_\ell)\in\{0,\dots,m-1\}^\ell} (\rho(A_{d_1}\cdots A_{d_\ell}))^{1/\ell}. \tag{8}$$

(3) For an arbitrary collection of operators $\{A_0,\dots,A_{m-1}\}$ the following inequalities hold:

$$\max_{(d_1,\dots,d_\ell)\in\{0,\dots,m-1\}^\ell} (\rho(A_{d_1}\cdots A_{d_\ell}))^{1/\ell} \leqslant \hat\rho(A) \leqslant (\max_\ell A)^{1/\ell}, \tag{9}$$
$$\check\rho(A) \leqslant \min_{(d_1,\dots,d_\ell)\in\{0,\dots,m-1\}^\ell} (\rho(A_{d_1}\cdots A_{d_\ell}))^{1/\ell} \leqslant (\min_\ell A)^{1/\ell}. \tag{10}$$

The proofs can be found in [11] and [12].

§ 3. Operators with invariant cone

Using the joint and the lower spectral radii we can find estimates of $\max_\ell A$ and $\min_\ell A$. These estimates are often crude, but they are quite satisfactory under certain assumptions about the operators. For instance, [10] considers the case of an irreducible collection of operators (that is, having no common non-trivial invariant subspaces). Stronger assumptions about the operators were put forward in [13], [14]. We consider here another case; namely, we assume that the operators have a common invariant cone.

Definition 2. A subset K of the Euclidean space $\mathbb{R}^s$ is called a *convex closed non-degenerate cone* (or simply a *cone* in what follows) if

(a) $x+y \in K$ for all $x, y \in K$;
(b) for each $x \in K \setminus \{0\}$ and each real coefficient λ the point λx lies in K if and only if $\lambda \geqslant 0$;
(c) K is a closed subset of $\mathbb{R}^s$ of dimension s, which means that there exists a ball $E(a,\varepsilon) = \{a + x : \|x\| \leqslant \varepsilon\}$ lying in K.

Definition 3. A cone $K \subset \mathbb{R}^s$ is called an *invariant cone* of a collection of operators $A = \{A_0, \dots, A_{m-1}\}$ if

$$AK = \bigcup_{i=0}^{m-1} A_i K \subset K. \tag{11}$$

We consider now several special properties of operators with invariant cone.

Lemma 1. *For each cone $K \subset \mathbb{R}^s$ and each norm in the space $\mathbb{R}^s$ there exists a constant μ depending on the cone and the norm such that for all $x, y \in K$ the 'reverse triangle inequality'*

$$\|x+y\| \geqslant \mu(\|x\| + \|y\|)$$

holds. For the Euclidean norm $\mu = \cos(\varphi/2)$, where φ is the largest angle between two vectors in K.

Proof. It suffices to consider the case of the Euclidean norm. Since K is a closed non-degenerate cone, it follows that $\varphi < \pi$. Hence

$$\|x+y\|^2 \geqslant \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| \cdot \cos\varphi \geqslant (\|x\| + \|y\|)^2 \cos^2 \frac{\varphi}{2}. \tag{12}$$

The proof is complete.

Lemma 2. *Let B be an operator with invariant cone K; then for each pair of vectors $t_1, t_2 \in K$,*

$$\|B(t_1)\| \cdot \|t_2\| \geqslant \mu h(t_1) \|B(t_2)\|, \tag{13}$$

where $h(t_1)$ is the distance from the point t_1 to the boundary of K.

Proof. For $t_2 = 0$ there is nothing to prove. If $t_2 \neq 0$, then there exists $\alpha > 0$ such that $t_1 \neq \alpha t_2$. We consider the point

$$y = t_1 + \frac{t_1 - \alpha t_2}{\|t_1 - \alpha t_2\|}\, h(t_1).$$

Clearly, $y \in K$. Applying Lemma 1 to the vectors $B(y)$ and $\dfrac{\alpha h(t_1)}{\|t_1 - \alpha t_2\|} B(t_2)$ we obtain

$$\begin{aligned}\left(1 + \frac{h(t_1)}{\|t_1 - \alpha t_2\|}\right)\|B(t_1)\| &= \left\| B(y) + \frac{\alpha h(t_1)}{\|t_1 - \alpha t_2\|} B(t_2) \right\| \\ &\geqslant \mu\left(\|B(y)\| + \frac{\alpha h(t_1)}{\|t_1 - \alpha t_2\|}\|B(t_2)\| \right) \geqslant \frac{\alpha \mu h(t_1)}{\|t_1 - \alpha t_2\|}\|B(t_2)\|.\end{aligned}$$

Thus,

$$\left(1+\frac{h(t_1)}{\|t_1-\alpha t_2\|}\right)\|B(t_1)\| \geqslant \frac{\alpha\mu h(t_1)}{\|t_1-\alpha t_2\|}\|B(t_2)\|. \tag{14}$$

Passing to the limit in (14) as $\alpha \to +\infty$ we obtain

$$\|B(t_1)\| \geqslant \frac{\mu h(t_1)}{\|t_2\|}\|B(t_2)\|,$$

which proves Lemma 2.

For $t \in K \setminus \{0\}$ let

$$\gamma(t) = \frac{h(t)}{\|t\|}\,. \tag{15}$$

Then we can write inequality (13) as follows:

$$\frac{\|B(t_1)\|}{\|t_1\|} \geqslant \mu\gamma(t_1)\frac{\|B(t_2)\|}{\|t_2\|}$$

for all $t_1, t_2 \in K \setminus \{0\}$.

We consider now a family of operators $A = \{A_0, \dots, A_{m-1}\}$ with invariant cone $K \subset \mathbb{R}^s$ and a point $x \in K$. Let $\max_\ell A(x)$ and $\min_\ell A(x)$ denote the following quantities:

$$\max_{(d_1,\dots,d_\ell)\in\{0,\dots,m-1\}^\ell} \|A_{d_1}\cdots A_{d_\ell}(x)\|, \qquad \min_{(d_1,\dots,d_\ell)\in\{0,\dots,m-1\}^\ell} \|A_{d_1}\cdots A_{d_\ell}(x)\|.$$

We must find estimates of them for each positive integer ℓ. Some results in this direction are available in [10], where we consider the general case (not assuming the existence of an invariant cone). We now require better estimates. To begin with we recall several definitions.

Definition 4. A subspace $L \subset \mathbb{R}^s$ of dimension $s-1$ is called a *support plane* of a cone K if $L \cap K \neq \{0\}$, but $L \cap \operatorname{int} K = \varnothing$.

Definition 5. A subspace L of $\mathbb{R}^s$ is called a *boundary plane* of a cone K if it is the linear hull of the intersection of this cone and some support plane of it.

Consider now a subset M of the boundary of a cone $K \subset \mathbb{R}^s$. Assume that M lies in some boundary plane of the cone. We call the intersection of the various boundary planes containing M the *minimal boundary plane* of M. For a fixed subset Y of $\mathbb{R}^s$ and $a \in \mathbb{R}^s$ the equality $\langle a, Y\rangle = 0$ means that $\langle a, x\rangle = 0$ for all $x \in Y$. The reader can easily prove for himself the following result.

Lemma 3. *Let $Y \subset \partial K \setminus \{0\}$ be a subset of some boundary plane of a cone K. Then a vector $y \in \mathbb{R}^s$ belongs to the minimal boundary plane of Y if and only if for each $a \in \mathbb{R}^s$ the condition*

$$\begin{cases} \langle a, Y\rangle = 0, \\ \langle a, K\rangle \leqslant 0 \end{cases} \tag{16}$$

yields the equality $\langle a, y\rangle = 0$.

Lemma 4. *Let $K \subset \mathbb{R}^s$ be an invariant cone of an operator B. We consider a non-empty subset Y of $\partial K \setminus \{0\}$ lying in some boundary plane of the cone K. Let M be the minimal boundary plane of Y. Then the condition $BY \subset Y$ yields the inclusion $BM \subset M$.*

Proof. Assume that $BY \subset Y$, but $BM \not\subset M$. Then there exists $y \in M$ such that $By \notin M$. By Lemma 3 there exists $a \in \mathbb{R}^s$ such that $\langle a, Y\rangle = 0$, $\langle a, K\rangle \leqslant 0$, but $\langle a, By\rangle \neq 0$. Then $\langle B^*a, Y\rangle = 0$, $\langle B^*a, K\rangle \leqslant 0$, and $\langle B^*a, y\rangle \neq 0$. However, $y \in M$, so that the last inequality is in contradiction with Lemma 3.

Lemma 5. *Let K be an invariant cone of an operator B. If $Bx = 0$ for some $x \in \operatorname{int} K$, then B is identically zero.*

Proof. If $B \not\equiv 0$, then we can select $y \in K$ and $\varepsilon \in \mathbb{R} \setminus \{0\}$ such that $By \neq 0$ and the points $x + \varepsilon y$ and $x - \varepsilon y$ lie in K, which contradicts the non-degeneracy of K.

Having finished with the preliminary work we can now proceed to the estimates of the quantities $\max_\ell A(x)$ and $\min_\ell A(x)$ in terms of $\hat\rho(A)$ and $\check\rho(A)$. First, we analyse the two special cases $\hat\rho(A) = 0$ and $\check\rho(A) = 0$.

Remark 1. In the proof of Lemmas 6 and 7 we shall use a special norm in $\mathbb{R}^s$ corresponding to the cone K. The unit sphere in this norm is the boundary of the set $\operatorname{conv}(S \cap K, -(S \cap K))$, where S is the unit Euclidean ball in $\mathbb{R}^s$. This norm has the following property: the norm of an operator with invariant cone K is attained at some vector in this cone.

Lemma 6. *Let $A = \{A_0, \dots, A_{m-1}\}$ be a collection of operators with invariant cone $K \subset \mathbb{R}^s$.*

(a) *If $\check\rho(A) = 0$, then the kernel of one of these operators contains a boundary plane of K.*

(b) *If $\hat\rho(A) = 0$, then the intersection of the kernels of $A_0, \dots, A_{m-1}$ contains a boundary plane of K.*

Proof. (a) If $A_i = 0$ for some i, then there is nothing to prove. Assume that $A_i \neq 0$ for all $i = 0, \dots, m-1$. Let L_i denote the linear hull of the set $\operatorname{Ker} A_i \cap \partial K$. Lemma 5 shows that

$$\operatorname{Ker} A_i \cap \operatorname{int} K = \{0\}.$$

Hence $L_i \cap \operatorname{int} K = \varnothing$. We now discuss the two possible cases.

(1) $L_i \neq \{0\}$ for some $i \in \{0, \dots, m-1\}$. Let M_i be the minimal boundary plane of L_i. Since $A_i L_i \subset L_i$, it follows by Lemma 4 that $A_i M_i \subset M_i$. If, moreover, $L_i = M_i$, then the proof is complete, while if $L_i \neq M_i$, then for some $y \in M_i$ we have $A_i y \neq 0$. Next, for arbitrary $P \subset \mathbb{R}^s$ we set $P^* = \{x \in \mathbb{R}^s : \langle x, P\rangle \leqslant 0\}$. For each $x \in (A_i K)^*$ we have $\langle A_i^* x, K\rangle \leqslant 0$ and, in addition, $\langle A_i^* x, L_i\rangle = \langle x, A_i L_i\rangle = 0$, because $L_i \subset \operatorname{Ker} A_i$. Hence $\langle A_i^* x, y\rangle = 0$ by Lemma 3, so that $0 = \langle A_i^* x, y\rangle = \langle x, A_i y\rangle$. Thus, $(A_i K)^* \subset (A_i y)^\perp$ and therefore $K^* \subset (A_i y)^\perp$. Hence $\operatorname{int} K^* = \varnothing$, which contradicts the non-degeneracy of the cone K.

(2) $L_i = \{0\}$ for each $i = 0, \dots, m-1$. In this case the quantity

$$\alpha_1 = \min_{\substack{x \in K,\ \|x\|=1 \\ i=0,\dots,m-1}} \|A_i x\|$$

is positive, therefore

$$\|A_{d_1}\cdots A_{d_\ell}y\| \geqslant \alpha_1^\ell\|y\|$$

for each $y\in K$ and each collection of indices $\{d_j\}\in\{0,\dots,m-1\}^\ell$. Hence we immediately obtain the equality $\check\rho(A)=\lim_{\ell\to\infty}(\min_\ell A)^{1/\ell}\geqslant\alpha_1>0$, which contradicts the condition $\check\rho(A)=0$.

(b) Let L denote the linear hull of the set $\partial K\cap\operatorname{Ker}A_0\cap\dots\cap\operatorname{Ker}A_{m-1}$. If $L\neq\{0\}$, then we can show as in case (a) that L coincides with its minimal boundary plane. On the other hand, if $L=\{0\}$, then

$$\beta_1=\max_{i=0,\dots,m-1}\ \min_{\substack{x\in K\\ \|x\|=1}}\|A_ix\|>0.$$

Hence for each $y\in K$ there exists $j\in\{0,\dots,m-1\}$ such that $\|A_jy\|\geqslant\beta_1y$. Consequently, $\max_\ell A(y)\geqslant\beta_1^\ell\|y\|$ for each $\ell\geqslant1$, so that $\hat\rho(A)\geqslant\beta_1$. This contradiction completes the proof of Lemma 6.

Lemma 7. *For an arbitrary collection of operators $A=\{A_0,\dots,A_{m-1}\}$ with invariant cone $K\subset\mathbb{R}^s$ there exist points $z_1,z_2,z_3,z_4\in K\setminus\{0\}$ such that for each $\ell\geqslant1$,*

(a) $\max_\ell A(z_1)\leqslant\hat\rho^\ell\|z_1\|$,
(b) $\max_\ell A(z_2)\geqslant\hat\rho^\ell\|z_2\|$,
(c) $\min_\ell A(z_3)\leqslant\check\rho^\ell\|z_3\|$,
(d) $\min_\ell A(z_4)\geqslant\check\rho^\ell\|z_4\|$.

Proof. If $\hat\rho=0$, then assertion (b) is obvious, and (a) is a consequence of Lemma 6. In the same way we can establish (c) and (d) for $\check\rho=0$. Thus, we shall assume in our discussion of (a) and (b) that $\hat\rho\neq0$, and in the discussion of (c) and (d) we assume that $\check\rho\neq0$. Let K_1 be the intersection of K with the unit sphere and let

$$\|A\|=\sup_{j=0,\dots,m-1}\|A_j\|.$$

(a) Assume the contrary: for each $x\in K\setminus\{0\}$ there exists $\ell\in\mathbb{N}$ and a collection $\{d_1,\dots,d_\ell\}\in\{0,\dots,m-1\}^\ell$ such that $\|A_{d_1}\cdots A_{d_\ell}(x)\|>\hat\rho^\ell\|x\|$. Next, for each $i\geqslant1$ we set

$$U_i=\{x\in K_1:\text{there exists }\ell\leqslant i\text{ such that }\max_\ell A(x)>\hat\rho^\ell\}.$$

We have

$$U_1\subset U_2\subset\cdots\quad\text{and}\quad\bigcup_{i=1}^\infty U_i=K_1.$$

The set K_1 is compact, therefore for large N we have

$$U_N=\bigcup_{i=1}^N U_i=K_1.$$

Hence

$$\beta_2 = \min_{x\in K_1} \max_{\ell\leqslant N}(\hat{\rho}^{-\ell} \max_{\ell} A(x)) > 1. \tag{17}$$

We consider now arbitrary $x_1 \in K \setminus \{0\}$. Using (17) we obtain a quantity $\ell_1 \leqslant N$ and a product of operators $\prod_{\ell_1} = A_{d_1} \cdots A_{d_{\ell_1}}$ such that

$$\Big\|\prod_{\ell_1}(x_1)\Big\| \geqslant \beta_2 \hat{\rho}^{\ell_1} \|x_1\|.$$

Setting $x_2 = \prod_{\ell_1} x_1$ we find $\ell_2 \leqslant N$ and a product $\prod_{\ell_2} = A_{d_{\ell_1+1}} \cdots A_{d_{\ell_1+\ell_2}}$ such that

$$\Big\|\prod_{\ell_2}(x_2)\Big\| \geqslant \beta_2 \hat{\rho}^{\ell_2} \|x_2\|.$$

Repeating this q times we obtain

$$\|x_q\| \geqslant \beta_2^{q-1} \hat{\rho}^{\ell_1+\cdots+\ell_{q-1}} \|x_1\|,$$

which shows that

$$\Big\|\prod_{\ell_{q-1}} \cdots \prod_{\ell_1}\Big\| \geqslant \beta_2^{q-1} \hat{\rho}^{\ell_1+\cdots+\ell_{q-1}}, \qquad \ell_k \leqslant N, \quad k = 1, \dots, q-1.$$

We raise both sides to the power $(\ell_1 + \cdots + \ell_{q-1})^{-1}$ and pass to the limit as $q \to \infty$ to obtain the inequality $\hat{\rho} \geqslant \beta_2^{1/N} \hat{\rho}$. This contradiction completes the proof of (a).

(b) Assume the contrary: for each point $x \in K \setminus \{0\}$ there exists $\ell \in \mathbb{N}$ such that $\max_\ell A(x) < \hat{\rho}^\ell \|x\|$. For arbitrary $i \geqslant 1$ we set

$$V_i = \{x \in K_1 : \text{there exists } \ell \leqslant i \text{ such that } \max_{\ell} A(x) < \hat{\rho}^\ell\}.$$

Clearly, $V_1 \subset V_2 \subset \cdots$ and, in addition, $\bigcup_{i=1}^\infty V_i = K_1$. Since K_1 is compact, there exists N such that

$$\gamma_1 = \max_{x\in K_1} \min_{\ell\leqslant N}\big(\hat{\rho}^{-\ell} \max_{\ell} A(x)\big) < 1. \tag{18}$$

Consider arbitrary $x_1 \in K \setminus \{0\}$, $j > N$, and a sequence $\{d_1, \dots, d_j\} \in \{0, \dots, m-1\}^j$. By the definition of γ_1 there exists $\ell_1 \leqslant N$ such that $\max_{\ell_1} A(x_1) \leqslant \gamma_1 \hat{\rho}^{\ell_1} \|x_1\|$. Hence

$$\|A_{d_{j-\ell_1+1}} \cdots A_{d_j} x_1\| \leqslant \gamma_1 \hat{\rho}^{\ell_1} \|x_1\|.$$

We now set $x_2 = A_{d_{j-\ell_1+1}} \cdots A_{d_j} x_1$ and find $\ell_2 < N$ such that

$$\max_{\ell_2} A(x_2) \leqslant \gamma_1 \hat{\rho}^{\ell_2} \|x_2\|.$$

Next, let $x_3 = A_{d_{j-\ell_1-\ell_2+1}} \cdots A_{j-\ell_1} x_2$, and so on, until at some ith step we obtain the inequality $\ell_1 + \ell_2 + \cdots + \ell_{i+1} > j$. Thus,

$$\|A_{d_1} \cdots A_{d_j} x_1\| \leqslant \|A_{d_1} \cdots A_{d_{j-\ell_1-\cdots-\ell_i}}\| \cdot \gamma_1^i \hat{\rho}^{\ell_1+\cdots+\ell_i} \|x_1\|.$$

Note now that $\|A_{d_1} \cdots A_{d_{j-\ell_1-\cdots-\ell_i}}\| \leqslant \max(1, \|A\|^N)$. On the other hand,

$$\hat{\rho}^{\ell_1+\cdots+\ell_i} \leqslant \frac{\hat{\rho}^j}{\min(1, \hat{\rho}^N)}\,.$$

Since $i \geqslant j/N - 1$, it follows that

$$\|A_{d_1} \cdots A_{d_j} x_1\| \leqslant \frac{\max(1, \|A\|^N)}{\min(1, \hat{\rho}^N)} \cdot \hat{\rho}^j \gamma_1^{j/N-1} \|x_1\|.$$

The same holds for each product of operators of length j and each point $x_1 \in K \backslash \{0\}$. Hence we may set $\|A_{d_1} \cdots A_{d_j}\| = \max_j A$ and moreover,

$$\|A_{d_1} \cdots A_{d_j} x_1\| = \|A_{d_1} \cdots A_{d_j}\| \cdot \|x_1\|$$

(see Remark 1). Thus,

$$\max_j A \leqslant \frac{\max(1, \|A\|^N)}{\min(1, \hat{\rho}^N)}\, \hat{\rho}^j \gamma_1^{j/N-1}.$$

Raising both sides to the power j^{-1} and passing to the limit as $j \to \infty$ we obtain the inequality $\hat{\rho} \leqslant \gamma_1^{1/N} \hat{\rho}$. This contradiction completes the proof of (b).

(c) If there exists $z \in K \setminus \{0\}$ such that for some $i \in \{0, \ldots, m-1\}$ we have $A_i z = 0$, then (c) holds for $z_3 = z$. If there exists no such point, then

$$a = \min_{\substack{i=0,\ldots,m-1 \\ x \in K_1}} \|A_i x\| > 0.$$

Assume now that (c) does not hold; then, as in the proof of (b), we can find $\ell, N \in \mathbb{N}$ and $\gamma_2 > 1$ such that $\ell \leqslant N$ and the inequality

$$\min_{\ell} A(x) \geqslant \gamma_2 \check{\rho}^{\ell} \|x\|$$

holds for each $x \in K \setminus \{0\}$. We now pick arbitrary $x_1 \in K \setminus \{0\}$ and a product $A_{d_1} \cdots A_{d_j}$ of length $j > N$. As in (b), we construct a sequence $x_1, x_2, \ldots, x_{i+1}$ such that

$$x_{k+1} = A_{d_{j-\ell_1-\cdots-\ell_k+1}} \cdots A_{d_{j-\cdots-\ell_{k-1}}} x_k, \qquad \|x_{k+1}\| \geqslant \gamma_2 \hat{\rho}^{\ell} \|x_k\|$$

for each $k=1,\dots,i$. Here $\ell_0=0$ and i is the largest integer such that $\ell_1+\dots+\ell_i \leqslant j$. Thus, $\|x_{i+1}\| \geqslant \gamma_2^i \check{\rho}^{\ell_1+\dots+\ell_i}\|x_1\|$. Hence

$$\|A_{d_1}\cdots A_{d_j}x_1\| = \|A_{d_1}\cdots A_{d_{j-\ell_1-\dots-\ell_i+1}}x_i\| \geqslant a^{j-\ell_1-\dots-\ell_i}\|x_i\| \\ \geqslant \gamma_2^i \check{\rho}^{\ell_1+\dots+\ell_{i-1}} a^{j-\ell_1-\dots-\ell_{i-1}}\|x_1\| \geqslant \frac{\min(1,a^N)}{\max(1,\check{\rho}^N)}\check{\rho}^j \gamma_2^{j/N-1}\|x_1\|.$$

The above inequality holds for each product of length j, therefore

$$\min_j Ax_1 \geqslant \frac{\min(1,a^N)}{\max(1,\check{\rho}^N)}\check{\rho}^j \gamma_2^{j/N-1}\|x_1\|.$$

Raising both sides to the power j^{-1} and passing to the limit as $j\to\infty$ we obtain the inequality $\check{\rho} \geqslant \gamma_2^{1/N}\check{\rho}$. This is a contradiction, which proves (c).

(d) Assume the contrary: for each $x\in K\setminus\{0\}$ there exists $\ell\in\mathbb{N}$ and a product $A_{d_1}\cdots A_{d_\ell}$ such that $\|A_{d_1}\cdots A_{d_\ell}x\| < \check{\rho}^\ell\|x\|$. We now repeat the proof of (a) replacing throughout "min", "$\check{\rho}$", and "$\geqslant$" by "max", "$\hat{\rho}$", and "$\leqslant$", respectively. For arbitrary $x_1\in K\setminus\{0\}$ we construct a sequence $x_1,x_2,\dots$ and products of operators $\prod_{\ell_1},\prod_{\ell_2},\dots$ such that for each $q\in\mathbb{N}$ we have

$$\Big\|\prod_{\ell_{q-1}}\cdots\prod_{\ell_1}x_1\Big\| \leqslant \beta_3^{q-1}\check{\rho}^{\ell_1+\dots+\ell_{q-1}}\|x_1\|,$$

where $\beta_3<1$. We shall assume without loss of generality that $h(x_1)>0$. Let $x_0\in K$ be a point such that

$$\Big\|\prod_{\ell_{q-1}}\cdots\prod_{\ell_1}x_0\Big\| = \Big\|\prod_{\ell_{q-1}}\cdots\prod_{\ell_1}\Big\|\cdot\|x_0\|$$

(see Remark 1). Applying Lemma 2 we obtain

$$\Big\|\prod_{\ell_{q-1}}\cdots\prod_{\ell_1}\Big\| = \frac{\|\prod_{\ell_{q-1}}\cdots\prod_{\ell_1}x_0\|}{\|x_0\|} \leqslant \frac{\|\prod_{\ell_{q-1}}\cdots\prod_{\ell_1}x_1\|}{\mu h(x_1)}.$$

Thus,

$$\Big\|\prod_{\ell_{q-1}}\cdots\prod_{\ell_1}\Big\| \leqslant \frac{1}{\mu h(x_1)}\beta_3^{q-1}\check{\rho}^{\ell_1+\dots+\ell_{q-1}}\|x_1\|.$$

We raise both sides to the power $(\ell_{q-1}+\dots+\ell_1)^{-1}$ and pass to the limit as $q\to\infty$. We obtain the inequality $\check{\rho}=\beta_3^{1/N}\check{\rho}$, contradicting the assumptions. The proof is complete.

Proposition 1. *Let* $A = \{A_0, \dots, A_{m-1}\}$ *be an operator family with invariant cone* $K \subset \mathbb{R}^s$. *Then for all* $x \in \operatorname{int} K$ *and* $\ell \in \mathbb{N}$,

(a) $\min_\ell A(x) \geqslant \check{\rho}^\ell \mu\gamma(x)\|x\|$;
(b) $\max_\ell A(x) \geqslant \hat{\rho}^\ell \mu\gamma(x)\|x\|$, *where the constant* $\mu = \mu(K)$ *has been defined in Lemma* 1;
(c) *if no common eigenspace of the operators* $A_0, \dots, A_{m-1}$ *is a boundary plane of the cone* K, *then there exists a constant* $H > 0$ *such that for each* $x \in K$,

$$\max_\ell A(x) \leqslant H\hat{\rho}^\ell \|x\|.$$

Proof. (a) Lemma 7 shows that there exists a point $z_3 \in K \setminus \{0\}$ such that for each $\ell \in \mathbb{N}$ we have

$$\min_\ell A(z_3) \geqslant \check{\rho}^\ell \|z_3\|.$$

Let $x \in \operatorname{int} K$ be arbitrary. Applying Lemma 2 to x and z_3 we obtain

$$\min_\ell A(x)\|z_3\| \geqslant \mu\gamma(x)\|x\| \min_\ell A(z_3) \geqslant \mu\gamma(x)\|x\|\check{\rho}^\ell \|z_3\|,$$

which proves (a). Assertion (b) can be proved in a similar way. We proceed to (c). Lemma 6 allows us to assume without loss of generality that $\hat{\rho} \neq 0$. Let U be the set of points $y \in \mathbb{R}^s$ such that the set $\{\hat{\rho}^{-\ell} \max_\ell A(y),\ \ell \in \mathbb{N}\}$ is bounded. Clearly, U is a common eigenspace of $A_0, \dots, A_{m-1}$. In addition, $U \cap K \neq \{0\}$ (as follows from Lemma 7(a)). Two cases are now possible.

(1) $U \cap \operatorname{int} K \neq \varnothing$. Lemma 2 shows that for all $y_0 \in U \cap \operatorname{int} K$ and $x \in K$ we have

$$\max_\ell A(y_0)\|x\| \geqslant \mu h(y_0) \max_\ell A(x).$$

Next, it follows from the definition of U that there exists a constant H such that $\max_\ell A(x) \leqslant H\hat{\rho}^\ell \|x\|$.

(2) $U \cap \operatorname{int} K = \varnothing$. In this case $U \cap K \subset \partial K$, therefore, using Lemma 7 we see that $U \cap \partial K \neq \{0\}$. Let M be the minimal boundary plane of $U \cap \partial K$. By Lemma 4, M is a common eigenspace of the operators $A_0, \dots, A_{m-1}$. This is a contradiction, which completes the proof of Proposition 1.

Remark 2. One more result suggests itself in the statement of Proposition 1, which could seem incomplete otherwise. Namely, one could conjecture that under certain conditions on the common eigenspaces of the operators $A_0, \dots, A_{m-1}$ there exists $H_1 > 0$ such that for all $\ell \in \mathbb{N}$ and $x \in K$ we have

$$\min_\ell A(x) \leqslant H_1 \check{\rho}^\ell \|x\|. \tag{19}$$

However, this is not true in the general case. Consider the following example.

Let A_0 and A_1 be two linear operators in $\mathbb{R}^2$:

$$A_0 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 1/2 \\ 1 & 1 \end{pmatrix}.$$

They are non-degenerate and have no common eigenspaces. The positive coordinate sector $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geqslant 0\}$ is a common invariant cone. Nevertheless, inequality (19) does not hold for these operators. Indeed, $\rho(A_0A_1) = 1$, so that $\check{\rho} \leqslant 1$ by (10). On the other hand, considering the norm $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$ in $\mathbb{R}^2$ we obtain

$$\|A_0x\|_1 \geqslant \|x\|_1, \qquad \|A_1x\|_1 \geqslant \|x\|_1 \quad \text{for all} \quad x \in K.$$

Hence $\check{\rho} \geqslant 1$ and therefore $\check{\rho} = 1$. Next, for each $x \in K$ we have

$$A_0^2x = 2x, \qquad \|A_1^2x\|_1 \geqslant \frac{3}{2}\|x\|_1, \tag{20}$$

therefore for each $x \in \operatorname{int} K$ and each sequence of zeros and ones $d_1d_2\dots$ we have

$$A_{d_1} \cdots A_{d_j}x \to \infty \quad \text{as } j \to \infty. \tag{21}$$

For if this sequence contains infinitely many pairs of the form $(0, 0)$ or $(1, 1)$, then (21) is a consequence of (20). On the other hand, for each $x \in \operatorname{int} K$ we have

$$(A_0A_1)^kx \to \infty \quad \text{as } j \to \infty.$$

Thus, (19) fails for A_0 and A_1.

Remark 3. The assumption about the common eigenspaces of $A_0, \dots, A_{m-1}$ is essential in part (b) of Proposition 1. For let us consider the 'collection' consisting of a single operator A_0 in $\mathbb{R}^2$:

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let K be the same invariant cone as in Remark 2. It is easy to see that $\hat{\rho} = 1$; however $A_0^{\ell}x \to \infty$ as $\ell \to \infty$ for each $x \in \operatorname{int} K$.

The growth estimate in Proposition 1(c) is not convenient for practical purposes because the constant H depends also on $A_0, \dots, A_{m-1}$ in addition to the invariant cone and can, generally speaking, be arbitrarily large (see [10]). Moreover, we do not have a similar lower bound for $\max_\ell A(x)$ in the general case (Remark 2). These difficulties can be overcome by means of a second invariant cone.

§ 4. Second invariant cone

Definition 6. A cone $K' \subset \mathbb{R}^s$ is said to be *embedded* in a cone K if the inclusion $(K' \setminus \{0\}) \subset \operatorname{int} K$ holds.

Let (K, K') be a pair of embedded cones (that is, K' is embedded in K). We set

$$\gamma = \gamma(K, K') = \inf_{x \in K' \setminus \{0\}} \frac{h(x)}{\|x\|}\,.$$

It is clear that $\gamma > 0$ and, in addition, $\gamma(x) \geqslant \gamma$ for each $x \in K'$.

Definition 7. A pair (K, K') is called a *pair of invariant cones* (an *invariant pair*) of a collection of operators $A = \{A_0, \dots, A_{m-1}\}$ if K and K' are invariant cones of this collection and K' is embedded in K.

Proposition 2. *If a collection of operators $A = \{A_0, \dots, A_{m-1}\}$ has an invariant pair (K, K'), then for all $x \in K'$ and $\ell \in \mathbb{N}$,*

$$\check{\rho}^{\ell} \mu\gamma \|x\| \leqslant \min_{\ell} A(x) \leqslant \check{\rho}^{\ell} (\mu\gamma)^{-1} \|x\|, \tag{22}$$

$$\hat{\rho}^{\ell} \mu\gamma \|x\| \leqslant \max_{\ell} A(x) \leqslant \hat{\rho}^{\ell} (\mu\gamma)^{-1} \|x\|. \tag{23}$$

Remark 4. Recall that the constant $\mu = \mu(K)$ was defined in Lemma 1. It depends on the cone K. In the case of the Euclidean norm $\mu = \cos(\varphi/2)$.

Proof of Proposition 2. The left-hand sides of (22) and (23) are consequences of Proposition 1. It remains to prove the right-hand sides. Applying Lemma 7 to the cone K' we obtain a point $z_3 \in K'$ such that

$$\min_{\ell} A(z_3) \leqslant \check{\rho}^{\ell} \|z_3\| \tag{24}$$

for each $\ell \in \mathbb{N}$. Consider now arbitrary $x \in K'$. Applying Lemma 2 to the points z_3, x, and the cone K we obtain

$$\mu\gamma(z_3)\|z_3\| \min_{\ell} A(x) \leqslant \|x\| \min_{\ell} A(z_3) \leqslant \check{\rho}^{\ell} \|z_3\| \cdot \|x\|.$$

Since $\gamma(z_3) \geqslant \gamma$, it follows that

$$\min_{\ell} A(x) \leqslant \check{\rho}^{\ell} (\mu\gamma)^{-1} \|x\|.$$

The inequality on the right-hand side of (23) can be established in a similar way.

We now present several examples of invariant pairs.

Example 1. Consider an arbitrary collection of operators $A = \{A_0, \dots, A_{m-1}\}$ with invariant cone K. Assume that for each $j = 0, \dots, m-1$ the set $A_j K \setminus \{0\}$ lies in the interior of K and let

$$\overline{A} K = \operatorname{conv}(A_0 K, \dots, A_{m-1} K).$$

It is easy to establish the existence of a cone K' in K such that $\overline{A} K \subset K'$. Hence (K, K') is an invariant pair of the collection A.

Example 2. Let $A_0, \dots, A_{m-1}$ be matrices with positive entries. We shall take the positive coordinate sector $K = \{(x_1, \dots, x_s) \in \mathbb{R}^s \colon x_i \geqslant 0,\ i = 1, \dots, s\}$ for the outer invariant cone; it then follows from the previous example that there exists an invariant cone K' embedded in K. Hence operators whose matrices contain only positive entries always have an invariant pair. Note, however, that the collection $\{A_0, A_1\}$ in Remark 2 possesses no invariant pairs, although the entries in the matrices of A_0 and A_1 are non-negative.

Example 3. Theorem 2.1 in [14] provides sufficient conditions for the existence of an invariant pair in the case of stochastic matrices.

§ 5. Asymptotic behaviour of the partition function

We shall now use the above results in estimates of the function $b_{m,d}(n)$. We shall prove that for each pair (m,d) there exist positive constants α_1, α_2, α_3, and α_4 dependent on m and d such that for each integer $\ell > 1 + \log_m d$,

$$\alpha_1 \leqslant \max_{m^{\ell-1} \leqslant n < m^\ell} b(n) n^{-\lambda_2} \leqslant \alpha_2 m, \tag{25}$$

$$\alpha_3 \leqslant \min_{m^{\ell-1} \leqslant n < m^\ell} b(n) n^{-\lambda_1} \leqslant \alpha_4 m, \tag{26}$$

where λ_1 and λ_2 are defined in (3). We shall also show that $\lambda_1 = \log_m \check{\rho}$ and $\lambda_2 = \log_m \hat{\rho}$, where $\check{\rho}$ and $\hat{\rho}$ are the lower spectral radius and the common spectral radius of an appropriate collection of linear operators. We shall calculate these quantities for some pairs (m,d) and find estimates for other pairs. We shall also obtain estimates of α_1, α_2, α_3, and α_4.

Let (m,d) be a pair of positive integers such that $2 \leqslant m \leqslant d-1$. We set

$$d = km + r, \tag{27}$$

where $k \in \mathbb{N}$ and $r \in \{0, \dots, m-1\}$. For arbitrary positive integer n and arbitrary $t_0 \in \{0, \dots, m-1\}$ we consider the representation of $mn + t_0$ with radix m:

$$mn + t_0 = \sum_{j=0}^{\ell} a_j m^j, \qquad a_j \in \{0, \dots, d-1\}. \tag{28}$$

Since $a_0 \equiv t_0 \pmod{m}$, it follows that $a_0 = k_0 m + t_0$ for some integer $k_0 \geqslant 0$. Hence

$$n - k_0 = \sum_{j=1}^{\ell} a_j m^{j-1}. \tag{29}$$

Since $k_0 \in \{0, \dots, \lfloor (d-1-t_0)/m \rfloor\}$, the formulae (28) and (29) establish a one-to-one correspondence between the representations of the quantity $mn + t_0$ and the representations of $n, n-1, \dots, n - \lfloor (d-1-t_0)/m \rfloor$. In view of the relation

$$\left\lfloor \frac{d-1-t_0}{m} \right\rfloor = \begin{cases} k, & t_0 < r, \\ k-1, & t_0 \geqslant r, \end{cases}$$

we obtain the recursive formula

$$b(mn + t_0) = \begin{cases} b(n) + \dots + b(n-k+1), & t_0 \geqslant r, \\ b(n) + \dots + b(n-k), & t_0 < r, \end{cases} \tag{30}$$

which can be written also as a matrix equality:

$$\begin{pmatrix} b(mn+m-1) \\ b(mn+m-2) \\ \vdots \\ b(mn+r) \\ b(mn+r-1) \\ \vdots \\ b(mn) \\ b(m(n-1)+m-1) \\ \vdots \\ \vdots \\ \vdots \\ b(m(n-1)) \\ b(m(n-2)+m-1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & \dots & 1 & 0 & 0 & 0 & \dots\dots \\ 1 & \dots & 1 & \vdots & \vdots & \vdots & \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \\ \vdots & & 1 & 0 & \vdots & \vdots & \\ \vdots & & 1 & 1 & \vdots & \vdots & \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \\ 1 & \dots & 1 & 1 & \vdots & \vdots & \\ 0 & 1 & \dots & 1 & \vdots & \vdots & \\ \vdots & \vdots & & \vdots & 0 & \vdots & \dots\dots \\ \vdots & \vdots & & \vdots & 1 & \vdots & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 1 & \dots & 1 & 1 & \vdots & \\ 0 & 0 & 1 & \dots & 1 & \vdots & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ \vdots & \vdots & 1 & \dots & 1 & 0 & \dots\dots \\ \vdots & \vdots & 1 & \dots & 1 & 1 & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} b(n) \\ b(n-1) \\ \vdots \\ b(n-k+1) \\ b(n-k) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \dots \end{pmatrix}. \tag{31}$$

Let M be the infinite matrix on the right-hand side of (31) and let $(M)_{ij}$ be the entry in its ith row and its jth column, $i, j \in \mathbb{N}$. The matrix M consists of equal $(m \times (k+1))$-blocks of the following form:

$$\begin{array}{r} m-r \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ r \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array} \begin{vmatrix} 1 & \dots & \dots & 1 & 0 \\ \vdots & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots \\ 1 & \dots & \dots & 1 & 0 \\ 1 & \dots & \dots & \dots & 1 \\ \vdots & & & & \vdots \\ 1 & \dots & \dots & \dots & 1 \end{vmatrix}.$$

The first $(m-r)$ rows have the form $(1, \dots, 1, 0)$ (k ones and one zero). The remaining rows have the form $(1, \dots, 1)$ ($k+1$ ones). Thus, there are d ones and $m-r$ zeros in the block. It is located in the first m rows and the first $k+1$ columns of M. The remaining entries of these rows are zeros. The next block lies in the rows with indices $m+1, \dots, 2m$ and in columns 2 through $(k+2)$. The remaining entries in rows $m+1, \dots, 2m$ of M are zeros. The next block lies m rows down and one column to the right, and so on.

We now split equality (31) into m linear relations in the space $\mathbb{R}^s$, where s is as follows:

$$s = k + \left\lceil \frac{k+r-1}{m-1} \right\rceil. \tag{32}$$

We consider the linear operators $A_0, \dots, A_{m-1}$ in $\mathbb{R}^s$ the matrices of which have the following entries:

$$(A_t)_{ij} = (M)_{i+m-t-1,j}, \qquad i, j \in \{1, \dots, s\}, \qquad t \in \{0, \dots, m-1\}. \tag{33}$$

This relation can be easily visualized: the matrix A_t is located in the first m columns and in rows $m-t, \dots, m-t+s-1$ of M. The reader can easily prove for himself that each row of the matrix of A_t contains a sequence of k or $k+1$ ones; its remaining components are equal to zero.

Consider now the vector-valued function

$$v(n) = v_{m,d}(n) = (b(n), \dots, b(n-s+1))^T, \qquad \text{where} \qquad b(n) = \begin{cases} b_{m,d}(n), & n \geqslant 0, \\ 0, & n < 0. \end{cases} \tag{34}$$

In particular, $v(0) = (1, 0, \dots, 0)^T$. Formulae (30) and (31) are equivalent to the m equalities

$$v(mn+t) = A_t v(n), \qquad t = 0, \dots, m-1. \tag{35}$$

We have thus established the following result.

Lemma 8. *For each pair (m, d), where $2 \leqslant m \leqslant d-1$, the function $v(n) = v_{m,d}(n)$ can be calculated by the following formula*:

$$v(n) = A_{t_0} \cdots A_{t_{\ell-1}} v(0),$$

where $t_{\ell-1}, \dots, t_0$ are the digits in the (radix) representation of n in the number system with base m:

$$n = \sum_{j=0}^{\ell-1} t_j m^j, \qquad t_j \in \{0, \dots, m-1\}.$$

Lemma 8 provides a precise formula for the partition function $b_{m,d}(n)$ for each pair m and d. We can now formulate our main result.

Theorem 1. *For each pair (m, d), where $2 \leqslant m \leqslant d-1$, there exist positive constants α_1, α_2, α_3, and α_4 dependent on m and d such that for each $\ell > 1 + \log_m d$,*

$$\alpha_1 \hat{\rho}^\ell \leqslant \max_{m^{\ell-1} \leqslant n < m^\ell} b(n) \leqslant \alpha_2 \hat{\rho}^\ell, \tag{36}$$

$$\alpha_3 \check{\rho}^\ell \leqslant \min_{m^{\ell-1} \leqslant n < m^\ell} b(n) \leqslant \alpha_4 \check{\rho}^\ell, \tag{37}$$

where $\hat{\rho} = \hat{\rho}(A_0, \dots, A_{m-1})$, $\check{\rho} = \check{\rho}(A_0, \dots, A_{m-1})$, and the operators $A_0, \dots, A_{m-1}$ are defined for fixed parameters (m, d) by formulae (31) *and* (33).

Moreover, the quantities $\hat{\rho}$ and $\check{\rho}$ satisfy the inequalities

$$\left\lfloor \frac{d}{m} \right\rfloor \leqslant \check{\rho} \leqslant \frac{d}{m} \leqslant \hat{\rho} \leqslant \left\lceil \frac{d}{m} \right\rceil. \tag{38}$$

Before proving the theorem we shall state several consequences of it.

Corollary 1. *Inequalities* (25) *and* (26) *hold for each* $\ell > 1 + \log_m d$.

Corollary 2. *For each pair* (m, d), *where* $2 \leqslant m \leqslant d-1$, *the following relations hold*:

$$\overline{\lim_{n\to\infty}} \frac{\log b(n)}{\log n} = \lambda_2 = \log_m \hat{\rho}(A_0, \dots, A_{m-1}),$$
$$\underline{\lim_{n\to\infty}} \frac{\log b(n)}{\log n} = \lambda_1 = \log_m \check{\rho}(A_0, \dots, A_{m-1});$$
$$\alpha_1 \leqslant \overline{\lim_{n\to\infty}}\, b(n) n^{-\lambda_2} \leqslant \alpha_2 m,$$
$$\alpha_3 \leqslant \underline{\lim_{n\to\infty}}\, b(n) n^{-\lambda_1} \leqslant \alpha_4 m.$$

Corollary 3. *For each* $n > md$,

$$\alpha_3 n^{\lambda_1} \leqslant b(n) \leqslant \alpha_2 m\, n^{\lambda_2}.$$

Remark 5. In the proof of Theorem 1 we shall obtain estimates of the quantities α_1, α_2, α_3, and α_4.

Proof of Theorem 1. Let $K = \{(x_1, \dots, x_s) \in \mathbb{R}^s : x_i \geqslant 0,\ i = 1, \dots, s\}$ be the positive coordinate sector in $\mathbb{R}^s$. Since all entries of the matrices of $A_0, \dots, A_{m-1}$ are non-negative, K is an invariant cone of this collection. The constant $\mu = \mu(K)$ is equal in this case to $\cos(\varphi/2) = \sqrt{2}/2$ (Lemma 1). As we shall see below, the existence of the second invariant cone depends on the relation between m and d. For some pairs (m, d) there exists an inner invariant cone K' and we can apply Proposition 2. For other pairs there is no inner cone, and we shall apply in that case the results of § 3.

We shall say that $x_1 \geqslant x_2$ for a pair of vectors $x_1, x_2 \in \mathbb{R}^s$ if $x_1 - x_2 \in K$. In a similar way, for operators B_1 and B_2 in $\mathbb{R}^s$ we shall say that $B_1 \geqslant B_2$ if $(B_1 - B_2)K \subset K$ (that is, the entries of the matrix of $B_1 - B_2$ are non-negative). It is easy to see that if the entries of B_1, B_2 and the components of x_1, x_2 are non-negative, then

$$x_1 \geqslant x_2,\ B_1 \geqslant B_2 \quad \Rightarrow \quad B_1 x_1 \geqslant B_2 x_2. \tag{39}$$

This means, in particular, that the common spectral radius and the lower spectral radius are non-decreasing functions of matrices with non-negative coefficients.

We now proceed to the proof of Theorem 1. Assume first that the operators $A_0, \dots, A_{m-1}$ have no inner invariant cone.

Case 1. $\boldsymbol{d \leqslant 2m-1}$**.** In this case $k = \lfloor d/m \rfloor = 1$, therefore $s = k + \left\lceil \dfrac{k+r-1}{m-1} \right\rceil = 2$. From (33) we deduce that $A_r = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Applying Lemma 8 to $n_\ell = \sum_{t=0}^{\ell-1} r m^t$ we obtain $v(n_\ell) = A_r^\ell v(0) = (1, \ell)^T$. Hence $b(n_\ell) = 1$ for each $\ell \in \mathbb{N}$, therefore

$$\min_{m^{\ell-1} \leqslant n < m^\ell} b(n) = 1, \qquad \ell \in \mathbb{N},$$

so that $\check{\rho} = 1$, $\lambda_1 = 0$, $\alpha_3 = \alpha_4 = 1$.

We must now consider two subcases.

(a) $m+2 \leqslant d \leqslant 2m-1$. Since $r \geqslant 2$, it follows that $A_{r-1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Hence $A_{r-1} \geqslant A_i$ for $i = 0, \dots, m-1$. Lemma 8 and relation (39) show that

$$v(n) = A_{t_0} \cdots A_{t_{\ell-1}} v(0) \leqslant A_{r-1}^{\ell} v(0) = (2^{\ell-1}, 2^{\ell-1})^T.$$

Thus,

$$\max_{m^{\ell-1} \leqslant n < m^{\ell}} b(n) = 2^{\ell-1}.$$

Hence $\hat{\rho} = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$.

(b) $d = m+1$.

In this subcase

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The other matrices in our collection are smaller than A_1. For if $m \geqslant 3$, then

$$A_{m-1} = \cdots = A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence for each positive integer ℓ and all $(t_0, \dots, t_{\ell-1}) \in \{0, \dots, m-1\}^{\ell}$ we have

$$A_{t_0} \cdots A_{t_{\ell-1}} \leqslant A_{\widetilde{t}_0} \cdots A_{\widetilde{t}_{\ell-1}},$$

where $\widetilde{t}_k = 0$ if $t_k = 0$, and $\widetilde{t}_k = 1$ otherwise. We obtain

$$\max_{m^{\ell-1} \leqslant n < m^{\ell}} b_{m,m+1}(n) = \max_{2^{\ell-1} \leqslant n < 2^{\ell}} b_{2,3}(n).$$

We have thus reduced the case $d = m+1$ to the case $m = 2$, $d = 3$ discussed by Reznick [2]. He showed, in particular, that

$$\max_{2^{\ell-1} \leqslant n < 2^{\ell}} b_{2,3}(n) = u_{\ell},$$

where $\{u_j\}$ is the Fibonacci sequence: $u_0 = u_1 = 1$, $u_{j+1} = u_j + u_{j-1}$. Using the well-known formula

$$u_j = \frac{\sqrt{5}}{5} \left(\left(\frac{\sqrt{5}+1}{2} \right)^{j+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{j+1} \right)$$

we conclude that

$$\max_{m^{\ell-1} \leqslant n < m^{\ell}} b_{m,m+1}(n) = u_{\ell} = \frac{5+\sqrt{5}}{10} \left(\frac{\sqrt{5}+1}{2} \right)^{\ell} + o(1) \quad \text{as } \ell \to \infty.$$

Hence $\hat{\rho} = (\sqrt{5}+1)/2$. This completes the proof for $d \leqslant 2n-1$. Note now that the collection of operators $\{A_0, \dots, A_{m-1}\}$ does not have a pair of invariant cones in this case because already the collection

$$\{A_0, A_r\} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

has no such pair. We shall see in what follows that in all remaining cases (that is, for $d \geqslant 2m$) there exists an invariant pair. In what follows we shall take for the outer invariant cone K the positive coordinate sector in $\mathbb{R}^s$. However, the inner cone K' will be different for different cases.

Case 2. $\boldsymbol{d = km,\ k \geqslant 2.}$ Here $r = 0$, therefore the sum of entries in each row of each matrix $A_0, \dots, A_{m-1}$ is k, that is, the matrices $k^{-1}A_0, \dots, k^{-1}A_{m-1}$ are stochastic with respect to the rows. Hence $\hat{\rho} = \check{\rho} = k$, so that for $d = km$ all the inequalities (38) become equalities.

For arbitrary $\beta > 1$ we consider now the cone

$$K_\beta = \bigl\{(x_1, \dots, x_s) \in K : \max_{i=1,\dots,s} x_i \leqslant \beta \cdot \min_{i=1,\dots,s} x_i\bigr\}.$$

It is an invariant cone of each matrix that is stochastic with respect to the rows. Hence the operators $\{A_0, \dots, A_{m-1}\}$ have in the present case a continuum of invariant pairs $\{(K, K_\beta),\ \beta \in (1, +\infty)\}$. For the constant $\gamma = \gamma(K, K_\beta)$ (see § 4) we obtain

$$\gamma(K, K_\beta) = \inf_{x \in K_\beta \setminus \{0\}} \frac{h(x)}{\|x\|} = \inf_{x \in K_\beta \setminus \{0\}} \min_{i=1,\dots,s} \frac{x_i}{\|x\|} \geqslant \frac{1}{\beta\sqrt{s}}. \tag{40}$$

(Recall that the norm in $\mathbb{R}^s$ is Euclidean.)

Before estimating the quantities $\alpha_1, \dots, \alpha_4$ we shall prove the existence of invariant pairs in the remaining cases.

Case 3. $\boldsymbol{d \geqslant 2m+1,\ r \neq 0,\ m \geqslant 3.}$ In this case K_β is an invariant cone for each $\beta \geqslant 2$. Recall that the rows in each matrix A_j contain sequences of either k or $k+1$ ones; we shall say that each row is either a k-*row* or a $(k+1)$-*row*. We shall now require the following auxiliary result:

In Case 3 *each k-row of the matrix A_t has at least two ones in common* (*that is, ones located in the same columns*) *with each $(k+1)$-row of A_t.*

For a proof we observe first of all that in Case 3 the inequality

$$\left\lceil \frac{k+r-1}{m-1} \right\rceil \leqslant k-1$$

holds for all k and r except for $k = 2$ and $r = m-1$. This inequality shows that $s \leqslant 2k-1$, so that each k-row and each $(k+1)$-row have at least two ones in common. In the remaining case (when $k = 2$ and $r = m-1$) the proof is carried out by direct verification.

We consider now an arbitrary vector $x \in K_\beta \setminus \{0\}$. We claim that $y = A_j x \in K_\beta$. Since x has positive components, we shall assume that $\min_{i=1,\dots,s} x_i = 1$. Hence $\max_{i=1,\dots,s} x_i \leqslant \beta$. Let y_t and y_q be the smallest and largest components (of the vector y), respectively. Clearly, $y_t \geqslant k$. If the qth row of A_j is a k-row, then $y_q \leqslant \beta k$, so that $y \in K_\beta$. If on the other hand it is a $(k+1)$-row, then it has at least two ones in common with the row y_t. We denote by i_1 and i_2 the indices of the rows containing these ones. We have $y_q \leqslant x_{i_1} + x_{i_2} + \beta(k-1)$. On the other hand $y_t \geqslant x_{i_1} + x_{i_2} + k - 2$. Hence $\beta y_t \geqslant y_q$, and therefore $y \in K_\beta$. Thus, $A_j K_\beta \subset K_\beta$, as required.

Case 4. $\boldsymbol{m = 2,\ d = 2k+1.}$ This is the last and the most complicated case. The point is that the cones K_β are no longer invariant for any β. One must look for an invariant cone of a more complex structure.

Since $m = 2$, it follows that $s = d - 1 = 2k$. The collection A consists now of two matrices:

$$(A_0)_{ij} = \begin{cases} 1 & \text{if } 1 \leqslant 2j - i \leqslant d, \\ 0 & \text{otherwise,} \end{cases} \qquad (A_1)_{ij} = \begin{cases} 1 & \text{if } 0 \leqslant 2j - i \leqslant d - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$

The matrix A_0 contains $k + 1$ ones in each odd row and k ones in each even row; A_1 contains $k + 1$ ones in even rows and k ones in odd rows. We consider now the following family of integer vectors:

$$V = \{(x_1, \dots, x_{2k}) \in \mathbb{R}^{2k} : x_i \in \{1, 2, 3, 4\},\ i = 1, \dots, 2k\}.$$

We remove from it the two vectors $a = (4, \dots, 4, 1, \dots, 1)$ and $b = (1, \dots, 1, 4, \dots, 4)$ (both containing k ones and k fours) to obtain a new family V'. We set

$$K' = \left\{ \sum_{i=1}^{N} \mu_i x_i : \mu_i \geqslant 0,\ x_i \in V',\ i = 1, \dots, N;\ N \in \mathbb{N} \right\}.$$

We now claim that K' is an invariant cone of the collection $\{A_0, A_1\}$. We shall show that $A_0 K' \subset K'$; the inclusion $A_1 K' \subset K'$ can be established in a similar way.

Let $x \in V'$ be an arbitrary vector. We must prove that $y = A_0 x$ is a positive linear combination of vectors in V'. First of all, it is clear that $k \leqslant y_i \leqslant 4(k + 1)$ for each $i = 1, \dots, 2k$. If $\min_{i=1,\dots,2k} y_i \geqslant k + 1$, then each component y_i can be represented as a sum of $k + 1$ integers from the set $\{1, 2, 3, 4\}$, so that y is representable as the sum of $k + 1$ (not necessarily distinct) vectors in V. Consider now another case when some components $\{y_i\}$ are equal to k. If

$$\max_{i=1,\dots,2k} y_i \leqslant 4k, \tag{42}$$

then y is a sum of k vectors in V. We claim that inequality (42) holds in the case $\min_{i=1,\dots,2k} y_i = k$. First, let $y_{2k} = k$; then the last k components of x are ones: $x_{k+1} = \cdots = x_{2k} = 1$. Since $x \in V'$, it follows that $x \neq a$, so that at least one of the coordinates $x_1, \dots, x_k$ is less than 4. Each component y_j is a sum of at most $k + 1$ coordinates of x, one of which (namely, x_{k+1}) is equal to 1, while another must be less than 4. Hence $y_j \leqslant 1 + 3 + 4(k - 1) = 4k$.

On the other hand, if there exists $j < 2k$ such that $y_j = k$, then at least two components of x are equal to 1. In fact it follows from (41) that in all rows of A_0 but the last, their kth and $(k + 1)$th components are ones. Hence $x_k = x_{k+1} = 1$ and therefore $y_i \leqslant 1 + 1 + 4(k - 1) = 4k - 2$ for each i. Hence y is a sum of k vectors in V.

Thus, the image of each vector in V' is a sum of k or $k + 1$ vectors in V. It remains to show that this sum can be chosen so as not to contain the 'forbidden' vectors a and b.

We start with the case of a sum of k vectors. Let $x \in V'$ and $y = A_0 x = v_1 + \cdots + v_k$, where $v_i \in V$, $i = 1, \dots, k$. Clearly, $y_i \in [k, 4k]$ for all $i = 1, \dots, 2k$. We consider the components y_k and y_{k+1}. We can assume without loss of generality that $y_k \leqslant y_{k+1}$.

Now, if $y_{k+1} \leqslant 3k$, then each of the quantities y_k, y_{k+1} is a sum of k integers from the set $\{1,2,3\}$. Hence we can choose $v_1,\dots,v_k$ such that the kth and the $(k+1)$th components of these vectors belong to the set $\{1,2,3\}$. Consequently, none of the vectors v_i is equal to a or b.

The case $y_k \geqslant 2k$ can be discussed in a similar way. Namely, we can show that there exist vectors $v_1,\dots,v_k \in V$ such that $v_1+\dots+v_k = y$ and the kth and the $(k+1)$th components of these vectors belong to the set $\{2,3,4\}$. Hence none of these vectors is equal to a or b.

We consider now the remaining case of $y_k \leqslant 2k-1$ and $y_{k+1} \geqslant 3k+1$. We set $y_k = k+t_1$ and $y_{k+1} = 4k-t_2$. Clearly, $t_1,t_2 \leqslant k-1$. As usual, we denote the jth component of v_i by $(v_i)_j$. In our case we can select the vectors $v_1,\dots,v_k$ so that

$$(v_i)_k = \begin{cases} 2, & 1 \leqslant i \leqslant t_1, \\ 1 & \text{otherwise}, \end{cases} \qquad (v_i)_{k+1} = \begin{cases} 3, & k-t_2+1 \leqslant i \leqslant k, \\ 4 & \text{otherwise}. \end{cases}$$

If, in addition, $t_1+t_2 \geqslant k$, then $a,b \notin \{v_1,\dots,v_k\}$ and the proof is complete. Let us show that, indeed, we have $t_1+t_2 \geqslant k$. Since two arbitrary consecutive rows of A_0 coincide in all but maybe one component, we have $y_{k+1}-y_k \leqslant 4$, so that $(4k-t_2)-(k+t_1) \leqslant 4$. Hence $t_1+t_2 \geqslant 3k-4 \geqslant k$ because $k \geqslant 2$. We have thus completed the analysis of the case of k vectors.

Assume now that $y = A_0x$ is a sum of $k+1$ vectors in V: $y = v_1+\dots+v_{k+1}$. We must again remove the vectors a and b from this sum. Again, assume that $y_{k+1} \geqslant y_k$. The cases of $y_k \geqslant 2(k+1)$ and $y_{k+1} \leqslant 3(k+1)$ can be considered in the same way as the cases of $y_k \geqslant 2k$ and $y_{k+1} \leqslant 3k$, respectively. It remains to discuss the last case, when $y_k \leqslant 2(k+1)-1$ and $y_{k+1} \geqslant 3(k+1)+1$. We set $y_k = (k+1)+t_1$ and $y_{k+1} = 4(k+1)-t_2$. Since $t_1,t_2 \leqslant k$, we can assume that the vectors $v_1,\dots,v_k$ satisfy the following conditions:

$$\begin{aligned} (v_i)_k &= 2 \quad \text{for } i = 1,\dots,t_1, \\ (v_i)_{k+1} &= 3 \quad \text{for } i = (k+1)-t_2+1,\dots,k+1. \end{aligned}$$

Note now that $t_1+t_2 \geqslant k+1$, as follows from the inequality $y_{k+1}-y_k \leqslant 4$. One then proceeds in a similar way to the previous case.

We have thus proved that K' is an invariant cone of the collection $\{A_0,A_1\}$. Note that $V' \subset K_\beta$ for $\beta \geqslant 4$, and therefore $K' \subset K_\beta$. Thus,

$$\gamma(K,K') \geqslant \gamma(K,K_\beta) \geqslant \frac{1}{\beta\sqrt{s}} = \frac{1}{4\sqrt{2k}}\,.$$

We can now complete the proof of Theorem 1 in cases 2, 3, and 4. We have established that $A_0,\dots,A_{m-1}$ have an invariant pair (K,K') if and only if $d \geqslant 2m$. In addition, we have proved that $\gamma(K,K') \geqslant (\beta\sqrt{s}\,)^{-1}$, where $\beta = 1$ for $r = 0$, $\beta = 4$ for $(m,d) = (2,2k+1)$, and $\beta = 2$ otherwise. Thus, in all possible cases

$$\gamma \geqslant \frac{1}{4\sqrt{s}}\,. \tag{43}$$

We set $u = (1, \ldots, 1)^T \in \mathbb{R}^s$. Applying Lemma 8 we obtain

$$\begin{aligned} b(n) =& \langle v(n), v(0)\rangle = \langle A_{t_0} \cdots A_{t_{\ell-1}} v(0), v(0)\rangle \\ &\leqslant \langle A_{t_0} \cdots A_{t_{\ell-1}} u, v(0)\rangle \leqslant \|A_{t_0} \cdots A_{t_{\ell-1}} u\| \end{aligned}$$

(we use the Euclidean norm). Next, on the basis of Proposition 2 we obtain

$$\min_{m^{\ell-1} \leqslant n < m^\ell} b(n) \leqslant \min_{m^{\ell-1} \leqslant n < m^\ell} \|A_{t_0} \cdots A_{t_{\ell-2}} u\| \leqslant \frac{1}{\mu\gamma} \check{\rho}^{\ell-1} \|u\| = \frac{1}{\mu\gamma} \check{\rho}^{\ell-1} \sqrt{s}\,,$$
$$\max_{m^{\ell-1} \leqslant n < m^\ell} b(n) \leqslant \max_{m^{\ell-1} \leqslant n < m^\ell} \|A_{t_0} \cdots A_{t_{\ell-2}} u\| \leqslant \frac{1}{\mu\gamma} \hat{\rho}^{\ell-1} \|u\| = \frac{1}{\mu\gamma} \hat{\rho}^{\ell-1} \sqrt{s}\,.$$

We have established the right-hand inequalities in (36) and (37). Moreover, we have obtained the estimates

$$\alpha_2 \leqslant \frac{\sqrt{s}}{\mu\gamma\hat{\rho}}\,, \qquad \alpha_4 \leqslant \frac{\sqrt{s}}{\mu\gamma\check{\rho}}\,. \tag{44}$$

To prove the left-hand inequalities in (36) and (37) we observe that for each $n \geqslant s$ all components of $v(n)$ are not smaller than 1. Hence if $q = \lceil \log_m s \rceil$, then for all $t_0, \ldots, t_q \in \{0, \ldots, m-1\}^{q+1}$ we have

$$A_{t_0} \cdots A_{t_q} v(0) \geqslant u.$$

Since each integer $n > md$ has at least $q+1$ digits in its (radix) representation with base m, it follows that

$$b(n) = \langle v(n),\, v(0)\rangle = \langle A_{t_0} \cdots A_{t_{\ell-1}} v(0), v(0)\rangle \geqslant \langle \Pi_{l-q-1} u, v(0)\rangle \geqslant \gamma \|\Pi_{l-q-1} u\|,$$

where $\Pi_{l-q-1} = A_{t_0} \cdots A_{t_{l-q-2}}$ for $l \geqslant q+2$ and Π_{l-q-1} is the identity operator for $l < q+2$. Thus, for arbitrary $\ell > 1 + \log_m d$ we have

$$\min_{m^{\ell-1} \leqslant n < m^\ell} b(n) \geqslant \gamma \min_{m^{\ell-1} \leqslant n < m^\ell} \|\Pi_{l-q-1} u\| \geqslant \mu\gamma^2 \check{\rho}^{\ell-q-1} \sqrt{s}\,,$$
$$\max_{m^{\ell-1} \leqslant n < m^\ell} b(n) \geqslant \gamma \max_{m^{\ell-1} \leqslant n < m^\ell} \|\Pi_{l-q-1} u\| \geqslant \mu\gamma^2 \hat{\rho}^{\ell-q-1} \sqrt{s}\,.$$

We have thus established the left-hand inequalities in (36) and (37). Moreover,

$$\alpha_3 \geqslant \frac{\mu\gamma^2\sqrt{s}}{\check{\rho}^{q+1}}\,, \qquad \alpha_1 \geqslant \frac{\mu\gamma^2\sqrt{s}}{\hat{\rho}^{\,q+1}}\,, \tag{45}$$

where $q = \lceil \log_m s \rceil$.

We now discuss (38). For each $j = 0, \ldots, m-1$ the matrix A_j contains k or $k+1$ ones in each row. Adding a single one in each k-row we obtain a matrix A_j^+ with $k+1$ ones in each row. On the other hand, removing a single one from each $(k+1)$-row we obtain a matrix A_j^- containing k ones in each row. Clearly, $A_j^- \leqslant A_j \leqslant A_j^+$, and the matrices $k^{-1} A_j^-$ and $(k+1)^{-1} A_j^+$ are stochastic with respect to the rows.

Since both the joint spectral radius and the lower spectral radius of a stochastic matrix are equal to one, it follows that

$$\hat{\rho}(A_0,\dots,A_{m-1}) \leqslant \hat{\rho}(A_0^+,\dots,A_{m-1}^+) = k+1 = \left\lceil \frac{d}{m} \right\rceil,$$
$$\check{\rho}(A_0,\dots,A_{m-1}) \geqslant \check{\rho}(A_0^-,\dots,A_{m-1}^-) = k = \left\lfloor \frac{d}{m} \right\rfloor.$$

Next, the matrix $B = d^{-1}(A_0 + \cdots + A_{m-1})$ is also stochastic with respect to the rows. Since

$$\check{\rho}(A_0,\dots,A_{m-1}) \leqslant \frac{d}{m}\rho(B) \leqslant \hat{\rho}(A_0,\dots,A_{m-1}),$$

it follows that

$$\check{\rho}(A_0,\dots,A_{m-1}) \leqslant \frac{d}{m} \leqslant \hat{\rho}(A_0,\dots,A_{m-1}).$$

We have thus proved (38), which completes the proof of Theorem 1.

Estimates of $\boldsymbol{\alpha_1}$, $\boldsymbol{\alpha_2}$, $\boldsymbol{\alpha_3}$, $\boldsymbol{\alpha_4}$. Using (43) and setting $\mu = \sqrt{2}/2$ in (44) we obtain

$$\alpha_2 \leqslant \frac{\sqrt{s}}{\hat{\rho}\mu\gamma} \leqslant \frac{\sqrt{s}\,\sqrt{2}\,4\sqrt{s}}{\hat{\rho}} = \frac{4\sqrt{2}\,s}{\hat{\rho}},$$
$$\alpha_4 \leqslant \frac{\sqrt{s}}{\check{\rho}\mu\gamma} \leqslant \frac{\sqrt{s}\,\sqrt{2}\,4}{\check{\rho}} = \frac{4\sqrt{2}\,s}{\check{\rho}}.$$

Next, using the inequalities $\hat{\rho} \geqslant d/m$ and $\check{\rho} \geqslant \lfloor d/m \rfloor$ we obtain

$$\alpha_2 \leqslant \frac{4ms\sqrt{2}}{d}, \qquad \alpha_4 \leqslant \frac{4s\sqrt{2}}{\lfloor d/m \rfloor}, \tag{46}$$

where $s = k + \left\lceil \dfrac{k+r-1}{m-1} \right\rceil$. We substitute (43) in (45) and set $\mu = \sqrt{2}/2$:

$$\alpha_3 \geqslant \frac{1}{\sqrt{2}\,16\sqrt{s}\,\check{\rho}^{q+1}}, \qquad \alpha_1 \geqslant \frac{1}{\sqrt{2}\,16\sqrt{s}\,\hat{\rho}^{\,q+1}}.$$

Finally, substituting the inequalities $\check{\rho} \leqslant d/m$, $\hat{\rho} \leqslant \lceil d/m \rceil$, and setting $q = \lceil \log_m s \rceil$ we obtain

$$\alpha_3 \geqslant \frac{1}{16\sqrt{2s}} \left(\frac{d}{m}\right)^{-\lceil \log_m s \rceil - 1}, \tag{47}$$
$$\alpha_1 \geqslant \frac{1}{16\sqrt{2s}} \left\lceil \frac{d}{m} \right\rceil^{-\lceil \log_m s \rceil - 1}. \tag{48}$$

§ 6. Growth exponents for $m = 2$ and $d = 2k + 1$

As pointed out in the introduction, Reznick [2] has calculated the growth exponents λ_1 and λ_2 for $m = 2$ and $d = 3$ and posed the question of the values of these exponents (in the case $m = 2$) for other odd values of d. We have solved this problem 'in principle' by showing that $\lambda_1 = \log_2 \check{\rho}(A_0, A_1)$ and $\lambda_2 = \log_2 \hat{\rho}(A_0, A_1)$. The question of the calculation of $\hat{\rho}$ and $\check{\rho}$ for each odd $d \geqslant 5$ now suggests itself.

Conjecture 1. *For all pairs* $(2, 2k+1)$, $k \in \mathbb{N}$, *the equalities*

$$\hat{\rho}(A_0, A_1) = \max\{\rho(A_0), \sqrt{\rho(A_0 A_1)}\}, \tag{49}$$

$$\check{\rho}(A_0, A_1) = \min\{\rho(A_0), \sqrt{\rho(A_0 A_1)}\} \tag{50}$$

hold (*here* ρ *is the usual spectral radius*).

Generally speaking, the problem of the calculation of the common spectral radius and the lower spectral radius for an arbitrary collection of operators is extremely complicated and the known algorithms are very slow (see, for instance, [9]–[11]). If Conjecture 1 holds, then this problem (for our matrices A_0 and A_1) can be reduced to an (asymmetric) eigenvalue problem for $(2k \times 2k)$-matrices. The values of $\hat{\rho}$ and $\check{\rho}$ will in this case be zeros of polynomials of degree $2k$, which in addition have integer coefficients. In particular, this means that $\hat{\rho}(A_0, A_1)$ and $\check{\rho}(A_0, A_1)$ are algebraic numbers.

We do not know whether Conjecture 1 holds for all $k \geqslant 2$. We shall prove it for some values of k.

Theorem 2. *Equalities* (49) *and* (50) *hold for each pair* $(2, 2k+1)$ *with* $1 \leqslant k \leqslant 6$.

Before the proof we make several observations. The case $(m, d) = (2, 3)$ has in fact been discussed in [2]. We must consider the cases $(2, 5)$, $(2, 7)$, $(2, 9)$, $(2, 11)$, and $(2, 13)$. First we shall formulate and prove Lemma 9 and Proposition 3, which are possibly also of independent interest: they suggest a new approach to the calculation of the joint spectral radius and the lower spectral radius in some special cases. We shall apply these techniques to the operators A_0, A_1 for $k = 2, \dots, 6$. We discuss each value of k separately, but use the same method. It is highly probable that the same method can help to extend Theorem 2 to other values of k. However, we have not managed to prove the theorem *for all positive integers* k (which would indeed be a strong result).

We use in the proof the Kreĭn–Rutman theorem (see, for instance, [15]), which states that for an arbitrary operator B with invariant cone $K \subset \mathbb{R}^s$ there exists a vector $v \in K$ such that $Bv = \rho(B)v$. We shall call it a *maximum vector* of B. An operator can in general have several maximum vectors.

Lemma 9. *Let* B_0, B_1 *be operators with invariant cone* $K \subset \mathbb{R}^s$.

(a) *If there exist a maximum vector* $v_0 \in \operatorname{int} K$ *of* B_0 *and an invariant cone* $\widetilde{K}$ *of* B_0, B_1 *such that* $K \subset \widetilde{K}$ *and* $(B_0 - B_1)v_0 \in \widetilde{K}$, *then*

$$\hat{\rho}(B_0, B_1) = \rho(B_0).$$

(b) *If there exist a maximum vector* $v_1 \in \operatorname{int} K$ *of* B_1 *and an invariant cone* $\widetilde{K}$ *of* B_0, B_1 *such that* $K \subset \widetilde{K}$ *and* $(B_0 - B_1)v_1 \in \widetilde{K}$, *then*

$$\check{\rho}(B_0, B_1) = \rho(B_1).$$

Proof. (a) We consider the set $H_0 = \widetilde{K} \cap (v_0 - \widetilde{K})$. Clearly, H_0 is convex and $0 \in H_0$. If H_0 is also unbounded, then it contains a ray $\{ty,\ t \geqslant 0\}$ with $y \in \widetilde{K} \setminus \{0\}$. Hence $\{ty,\ t \geqslant 0\} \subset v_0 - \widetilde{K}$, so that $v_0/t - y \in \widetilde{K}$ for all $t > 0$. Consequently, $(-y) \in \widetilde{K}$, which contradicts the non-degeneracy of $\widetilde{K}$. Thus, H_0 is a bounded set. Since $B_i H_0 \subset \rho(B_0) H_0$ for $i = 0, 1$, the set

$$\bigl\{\rho^{-\ell}(B_0) \max_{\ell} B(x),\ \ell \in \mathbb{N}\bigr\} \tag{51}$$

is bounded for each $x \in H_0$. Note that H_0 has non-empty interior since $v_0 \in \operatorname{int} K$. Thus, we can assume without loss of generality that $x \in \operatorname{int} H_0$. Combining (51) and Proposition 1(b) we obtain the inequality $\rho(B_0) \geqslant \hat{\rho}(B_0, B_1)$. Finally, we can apply (9) to prove that $\rho(B_0) = \hat{\rho}(B_0, B_1)$.

(b) Note first that the set $H_1 = v_1 + \widetilde{K}$ does not contain the origin, for otherwise $(-v_1) \in \widetilde{K}$, which contradicts the non-degeneracy of $\widetilde{K}$. Since $B_i H_1 \subset \rho(B_1) H_1$ for $i = 0, 1$, the set

$$\bigl\{\rho^{-\ell}(B_1) \min_{\ell} B(x),\ \ell \in \mathbb{N}\bigr\}$$

is bounded below for each $x \in H_1$. Hence $\check{\rho}(B_0, B_1) \geqslant \rho(B_1)$. Taking account of (10) we now obtain the equality $\check{\rho}(B_0, B_1) = \rho(B_1)$, which completes the proof of Lemma 9.

One consequence of Lemma 9 is the following Proposition 3 describing sufficient conditions for the relations

$$\hat{\rho}(B_0, B_1) = \rho(B_0), \qquad \check{\rho}(B_0, B_1) = \rho(B_1). \tag{52}$$

Before stating it we recall some notation. For a cone $K \subset \mathbb{R}^s$, a vector $x \in \mathbb{R}^s$, and an operator B the relations $x \geqslant 0$ $(x > 0)$ and $B \geqslant 0$ $(B > 0)$ mean that $x \in K$ $(x \in \operatorname{int} K)$ and $BK \subset K$ $(B(K \setminus \{0\}) \subset \operatorname{int} K)$, respectively. In particular, if $K = \{(x_1, \dots, x_s) \in \mathbb{R}^s,\ x_i \geqslant 0,\ i = 1, \dots, s\}$, then the inequality $x \geqslant 0$ $(x > 0)$ means that x has non-negative (positive) components, $B \geqslant 0$ $(B > 0)$ means that the matrix of B has non-negative (positive) entries. Let I_s be the identity operator in $\mathbb{R}^s$.

Proposition 3. *Let* $K \subset \mathbb{R}^s$ *be a fixed cone and* $B_0, B_1 \geqslant 0$ *operators such that each* B_i *has a maximum vector* $v_i \in \operatorname{int} K$ $(i = 0, 1)$. *Then each of the following conditions is sufficient for relations* (52):

(1) *there exists* $r \in \mathbb{N}$ *such that*

$$B_{i_1} \cdots B_{i_r} (B_0 - B_1) v_{i_0} \geqslant 0$$

for all $(i_0, \dots, i_r) \in \{0, 1\}^{r+1}$;

(2) *there exist non-degenerate operators* R_0 *and* R_1 *and positive integers* r, q, *and* N *such that*

  (a) *for* $i \in \{0,1\}$ *either* $R_i = I_s$ *or* $R_i > 0$ *and in addition* $R_i^{-1} B_i^{\ell} R_i \geqslant 0$ *for* $\ell = q, \dots, 2q-1$;

  (b) $B_{i_1} \cdots B_{i_r}(B_0 - B_1) B_{i_0}^N R_{i_0} \geqslant 0$ *for all* $(i_0, \dots, i_r) \in \{0,1\}^{r+1}$;

(3) *there exist a non-degenerate operator* $P \geqslant 0$ *and a positive integer* N *such that*

$$P B_i P^{-1} \geqslant 0, \qquad P(B_0 - B_1) B_i^N \geqslant 0 \quad \textit{for } i = 0, 1.$$

Proof. Part (1) is an immediate consequence of Lemma 9, where

$$\widetilde{K} = \bigcap_{(i_1,\dots,i_r)\in\{0,1\}^r} (B_{i_1} \cdots B_{i_r})^{-1} K.$$

(2) If $R_i = I_s$, then we have $v_i \in B_i^N K = B_i^N R K$. Hence the inequality $B_{i_1} \cdots B_{i_r}(B_0 - B_1) B_i^N R_i \geqslant 0$ means that $B_{i_1} \cdots B_{i_r}(B_0 - B_1) v_i \geqslant 0$.

If $R_i > 0$ and $R_i^{-1} B_i^{\ell} R_i \geqslant 0$, $\ell = q, \dots, 2q-1$, then we consider the set $K_i = \operatorname{conv}(B_i^q R_i K, \dots, B_i^{2q-1} R_i K)$. Note first that $K_i \subset R_i K$. Since $R_i > 0$, it follows that K_i is a subset of K. Moreover, $B_i K_i \subset K_i$. Now, using the Kreĭn–Rutman theorem we conclude that B_i has a maximum vector $u_i \in K_i$. We can assume without loss of generality that $v_i = u_i$. It remains to observe that $v_i \in B_i^N K_i \subset B_i^N R_i K$. Thus, in both cases we have

$$B_{i_1} \cdots B_{i_r} (B_0 - B_1) v_i \geqslant 0.$$

We now use part (1) for the final step of the proof.

(3) Since the set $P^{-1}K$, which contains K, is an invariant cone of B_0 and B_1, we can apply Lemma 9 with $\widetilde{K} = P^{-1}K$. The proof is complete.

Proof of Theorem 2. As usual, let $\{e_i\}_{i=1}^{2k} = \{(0, \dots, 0, 1, 0, \dots, 0)^T\}$ be the basis vectors in $\mathbb{R}^{2k}$. For each $(a_1, \dots, a_{2k})^T \in \mathbb{R}^{2k}$ let $[a_1, \dots, a_{2k}]$ be the $(2k \times 2k)$-matrix whose ith row is $(a_i, \dots, a_i)$, $i = 1, \dots, 2k$. Finally, let M be the matrix such that $M e_j = e_{2k+1-j}$. It is easy to see that $M^2 = I_{2k}$.

Thus, we have two $(2k \times 2k)$-matrices A_0 and A_1 defined by relations (41) and having an invariant pair (K, K') described in the proof of Theorem 1. Note that $A_1 = M A_0 M$. This shows, in particular, that $\rho(A_1) = \rho(A_0)$. Moreover,

$$\sqrt{\rho(A_0 A_1)} = \sqrt{\rho(A_0 M A_0 M)} = \rho(A_0 M) = \rho(M A_0).$$

It is an immediate consequence of the definition of the joint spectral radius that

$$\hat{\rho}(A_0, A_1) = \hat{\rho}(A_0, M A_0 M) = \hat{\rho}(A_0, M A_0).$$

In a similar way $\check{\rho}(A_0, A_1) = \check{\rho}(A_0, M A_0)$. In fact, for all $k_1, \dots, k_n \in \mathbb{N}$ we have

$$\begin{aligned}
A_0^{k_1} A_1^{k_2} \cdots A_1^{k_{n-1}} A_0^{k_n} &= A_0^{k_1} (M A_0 M)^{k_2} \cdots (M A_0 M)^{k_{n-1}} A_0^{k_n} \\
&= A_0^{k_1} (M A_0) A_0^{k_2 - 1} (M A_0) \cdots (M A_0) A_0^{k_{n-1}-1} (M A_0) A_0^{k_n - 1}, \\
A_1^{k_1} A_0^{k_2} \cdots A_1^{k_{n-1}} A_0^{k_n} &= (M A_0 M)^{k_1} A_0^{k_2} \cdots (M A_0 M)^{k_{n-1}} A_0^{k_n} \\
&= (M A_0) A_0^{k_1 - 1} (M A_0) A_0^{k_2 - 1} \cdots (M A_0) A_0^{k_{n-1}-1} (M A_0) A_0^{k_n - 1}.
\end{aligned}$$

Hence an arbitrary product of several factors equal to A_0 or A_1, with last factor A_0, is a product (with the same number of factors) of several copies of the operators A_0 and MA_0. In a similar way one can prove that a product of several copies of A_0 and A_1, with last factor equal to A_1, is a product (with the same number of factors) of copies of A_0, MA_0, multiplied on the right by M. Hence we immediately obtain the inequality $\hat{\rho}(A_0, A_1) \leqslant \hat{\rho}(A_0, MA_0)$. In a similar way, $\hat{\rho}(A_0, A_1) \geqslant \hat{\rho}(A_0, MA_0)$.

Thus, equalities (49) and (50) take the following form:

$$\hat{\rho}(A_0, MA_0) = \max\{\rho(A_0),\, \rho(MA_0)\},$$
$$\check{\rho}(A_0, MA_0) = \min\{\rho(A_0),\, \rho(MA_0)\}.$$

By the construction of K' (see the proof of Theorem 1(4)), $MK' = K'$ and $MK = K$. Hence (K, K') is an invariant pair also for the collection $\{A_0, MA_0\}$. It is now easy to show that $\det A_0 = (-1)^k$, so that the operators A_0 and MA_0 are non-degenerate. We now want to use Proposition 3 with $\{B_0, B_1\} = \{A_0, MA_0\}$. However, none of its assumptions holds for this collection of operators and the cone K. This nuisance can be overcome by the passage to another basis in $\mathbb{R}^{2k}$. The transformation matrix T will be defined as follows: the jth row of T is

$$\begin{aligned}
&(e_{j+k} - e_j)^T, && \text{for } 1 \leqslant j \leqslant k,\\
&(e_k)^T, && \text{for } j = k+1,\\
&(e_{2k-j+1} - e_{3k-j+2})^T, && \text{for } k+2 \leqslant j \leqslant 2k.
\end{aligned}$$

The inverse matrix T^{-1} has the following form:

$$\text{for } i \leqslant k \qquad (T^{-1})_{ij} = \begin{cases} 1, & i+1 \leqslant j \leqslant 2k-i+1,\\ 0 & \text{otherwise,}\end{cases}$$
$$\text{for } i \geqslant k+1 \qquad (T^{-1})_{ij} = \begin{cases} 1, & i-k \leqslant j \leqslant 3k-i+1,\\ 0 & \text{otherwise.}\end{cases}$$

We set $F = T^{-1}A_0T$ and $G = T^{-1}MA_0T$. The reader can easily find the explicit form of F and G, and we are content with the fact that each matrix consists of zeros, ones, and twos. This shows, in particular, that the positive coordinate sector K is an invariant cone of F and G. Note finally that the cone $T^{-1}K'$ is embedded in K (because T^{-1} is a non-degenerate matrix with non-negative entries) and is also an invariant cone of F and G. Thus, the collection $\{F, G\}$ has an invariant pair $(K, T^{-1}K')$. By the Kreĭn–Rutman theorem the cone $T^{-1}K'$ contains maximum vectors of F and G. We can now apply Proposition 3. The easiest way would be to use its part (3): one merely has to produce a suitable matrix P. This is not always possible, however. We can show that if $\rho(F) > \rho(G)$, then the assumptions of part (3) do not hold. In that case we shall use part (2). To this end it suffices to find appropriate matrices R_0, R_1 and numbers r, q, and N. We are not going to write down all the matrix products for reasons of space. All our calculations are precise (because the matrices have integer entries) and can be easily verified, for instance, on a computer. For $d = 5$ we present the results of all our calculations, and in the other cases we merely write down the most important matrices.

$d = 5$. In this case we have the two matrices

$$A_0 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad MA_0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

We can now find the transformation matrix T and its inverse T^{-1}:

$$T = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Next, we calculate F and G:

$$F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

We can use Proposition 3(3) with $B_0 = G$ and $B_1 = F$. The matrix P is as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We verify the inequalities $PB_iP^{-1} \geqslant 0$, $i = 0, 1$, first. Indeed, we have

$$PB_0P^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad PB_1P^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

For $N = 7$ we have

$$P(B_0 - B_1)B_0^7 = \begin{pmatrix} 74 & 20 & 35 & 107 \\ 76 & 22 & 18 & 65 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P(B_0 - B_1)B_1^7 = \begin{pmatrix} 198 & 80 & 1 & 9 \\ 176 & 110 & 2 & 19 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\hat{\rho}(A_0, MA_0) = \hat{\rho}(F, G) = \rho(G) = \rho(MA_0),$$
$$\check{\rho}(A_0, MA_0) = \check{\rho}(F, G) = \rho(F) = \rho(A_0),$$

which completes the proof of Theorem 2 for $d = 5$.

$d = 7$. We have

$$A_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix};$$

$$T^{-1} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We can apply Proposition 3(2) with $B_0 = F$, $B_1 = G$, $R_0 = [13, 8, 2, 16, 11, 5] + 3I_6$, $R_1 = I_6$, $r = 3$, $q = 2$, $N = 10$. Recall that I_{2k} is the identity $(2k \times 2k)$-matrix and $[a_1, \dots, a_{2k}]$ is the $(2k \times 2k)$-matrix with ith row $(a_i, \dots, a_i)$, $i = 1, \dots, 2k$. After the verification of the following 18 inequalities:

$$R_0^{-1} B_0^2 R_0 \geqslant 0, \qquad R_0^{-1} B_0^3 R_0 \geqslant 0,$$
$$B_{i_1} B_{i_2} B_{i_3} (B_0 - B_1) B_{i_0}^{10} R_{i_0} \geqslant 0, \qquad (i_0, \dots, i_3) \in \{0, 1\}^4,$$

we obtain

$$\hat{\rho}(A_0, MA_0) = \hat{\rho}(F, G) = \rho(F) = \rho(A_0),$$
$$\check{\rho}(A_0, MA_0) = \check{\rho}(F, G) = \rho(G) = \rho(MA_0),$$

which completes the proof for $d = 7$.

$d = 9$. We apply Proposition 3(3) to the matrices $B_0 = G$, $B_1 = F$, and

$$P = \begin{pmatrix} 2 & 5 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 6 & 5 & 0 & 0 & 0 & 0 & 0 \\ 4 & 10 & 8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8 & 10 & 4 \\ 0 & 0 & 0 & 0 & 0 & 5 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 5 & 2 \end{pmatrix}.$$

We have

$$PB_0P^{-1} = \begin{pmatrix} 0&0&0&0&2&0&1&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&2&1&0&0\\ 0&0&0&0&4&0&0&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&0&0&1&1&1\\ 1&0&0&0&0&0&0&1\\ 1&0&1&0&0&1&0&1 \end{pmatrix}, \ PB_1P^{-1} = \begin{pmatrix} 0&1&0&2&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&2&0&0&0&0\\ 1&0&0&4&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 1&1&1&0&0&1&1&1\\ 1&0&0&0&0&0&0&1\\ 1&0&0&0&0&1&0&1 \end{pmatrix}.$$

We now set $N = 13$ and verify the inequalities

$$P(B_0 - B_1)\,B_0^{13} \geqslant 0,$$
$$P(B_0 - B_1)\,B_1^{13} \geqslant 0.$$

Hence

$$\hat{\rho}(A_0, MA_0) = \hat{\rho}(F, G) = \rho(G) = \rho(MA_0),$$
$$\check{\rho}(A_0, MA_0) = \check{\rho}(F, G) = \rho(F) = \rho(A_0).$$

$d = 11$. We apply Proposition 3(2) to the matrices $B_0 = F$, $B_1 = G$, $R_1 = I_{10}$, and $R_0 = [36, 29, 20, 12, 3, 41, 33, 24, 16, 8] + 5I_{10}$ and the integers $r = 3$, $q = 4$, and $N = 12$.

On verifying the four inequalities

$$R_0^{-1} B_0^{\ell} R_0 \geqslant 0, \qquad \ell = 4, 5, 6, 7,$$

and the 16 inequalities

$$B_{i_1} B_{i_2} B_{i_3} (B_0 - B_1) B_1^{12} \geqslant 0, \qquad B_{i_1} B_{i_2} B_{i_3} (B_0 - B_1) B_0^{12} R_0 \geqslant 0,$$
$$(i_1, i_2, i_3) \in \{0, 1\}^3,$$

we obtain

$$\hat{\rho}(A_0, MA_0) = \hat{\rho}(F, G) = \rho(F) = \rho(A_0),$$
$$\check{\rho}(A_0, MA_0) = \check{\rho}(F, G) = \rho(G) = \rho(MA_0).$$

$d = 13$. We apply Proposition 3(2) to the matrices $B_0 = F$, $B_1 = G$, $R_1 = I_{12}$, and $R_0 = [15, 12, 9, 6, 4, 1, 16, 13, 11, 8, 5, 2] + 7I_{12}$ and the integers $r = 3$, $q = 4$, and $N = 16$.

On verifying the four inequalities

$$R_0^{-1} B_0^{\ell} R_0 \geqslant 0, \qquad \ell = 4, 5, 6, 7,$$

and the 16 inequalities

$$B_{i_1} B_{i_2} B_{i_3} (B_0 - B_1) B_1^{16} \geqslant 0, \qquad B_{i_1} B_{i_2} B_{i_3} (B_0 - B_1) B_0^{16} R_0 \geqslant 0,$$

where $(i_1, i_2, i_3) \in \{0,1\}^3$, we obtain

$$\hat{\rho}(A_0, MA_0) = \hat{\rho}(F, G) = \rho(F) = \rho(A_0),$$
$$\check{\rho}(A_0, MA_0) = \check{\rho}(F, G) = \rho(G) = \rho(MA_0).$$

The proof of Theorem 2 is complete.

We can now find the values of $\hat{\rho}(A_0, A_1)$ and $\check{\rho}(A_0, A_1)$ for odd $d \leqslant 13$.

$d = 3$.

$$\hat{\rho}(A_0, A_1) = \sqrt{\rho(A_0 A_1)} = \frac{\sqrt{5}+1}{2} = 1.61803\ldots,$$
$$\check{\rho}(A_0, A_1) = \rho(A_0) = 1 \qquad \text{(see [2])}.$$

$d = 5$.

$$\hat{\rho}(A_0, A_1) = \sqrt{\rho(A_0 A_1)} = \operatorname{root}(z^4 - 2z^3 - 2z^2 + 2z - 1) = 2.53861\ldots,$$
$$\check{\rho}(A_0, A_1) = \rho(A_0) = \sqrt{2} + 1 = 2.41421\ldots.$$

(Here $\operatorname{root}(p(z))$ is the largest (in absolute value) zero of the polynomial $p(z)$. We use this notation only for polynomials with one such zero.)

$d = 7$.

$$\hat{\rho}(A_0, A_1) = \rho(A_0) = \frac{1}{6}(332 + 12\sqrt{321})^{1/3} + \frac{20}{3(332 + 12\sqrt{321})^{1/3}} + \frac{4}{3}$$
$$= 3.51154\ldots,$$
$$\check{\rho}(A_0, A_1) = \sqrt{\rho(A_0 A_1)} = \operatorname{root}(z^5 - z^4 - 7z^3 - 5z^2 - 3z - 1) = 3.49189\ldots.$$

$d = 9$.

$$\hat{\rho}(A_0, A_1) = \sqrt{\rho(A_0 A_1)} = \operatorname{root}(z^8 - 3z^7 - 9z^6 + 9z^5 + 5z^4 - z^3 - z^2 - z + 1)$$
$$= 4.50309\ldots,$$
$$\hat{\rho}(A_0, A_1) = \rho(A_0) = \frac{1}{6}(908 + 12\sqrt{993})^{1/3} + \frac{44}{3(908 + 12\sqrt{993})^{1/3}} + \frac{4}{3}$$
$$= 4.49449\ldots.$$

$d = 11$.

$$\hat{\rho}(A_0, A_1) = \rho(A_0) = \operatorname{root}(z^4 - 5z^3 - 3z^2 + z + 1) = 5.50589\ldots,$$
$$\check{\rho}(A_0, A_1) = \sqrt{\rho(A_0 A_1)} = \operatorname{root}(z^{10} - 4z^9 - 12z^8 + 20z^7 + 42z^6 - 1) = 5.49704\ldots.$$

$d = 13$.

$$\hat{\rho}(A_0, A_1) = \rho(A_0) = \operatorname{root}(z^6 - 8z^5 + 10z^4 - 2z^3 + 2z^2 - 1) = 6.50216\ldots,$$
$$\check{\rho}(A_0, A_1) = \sqrt{\rho(A_0 A_1)}$$
$$= \operatorname{root}(z^{12} - 4z^{11} - 20z^{10} + 20z^9 + 28z^8 + 4z^7 + 8z^6 + 4z^4 + 4z^3 - 1)$$
$$= 6.49894\ldots.$$

Theorem 2 enables us to find explicit sequences of integers n delivering the upper limit λ_2 and the lower limit λ_1 in (3). Consider the sequences $x_r = 4^r$ and $y_r = (4^{r+1} - 1)/3$. By Lemma 8 we obtain

$$v(x_r) = A_0^{2r} A_1 v(0), \qquad v(y_r) = (A_1 A_0)^r A_1 v(0),$$

where $v(n) = \bigl(b_{2,d}(n), \dots, b_{2,d}(n - 2k + 1)\bigr)^T$. Hence

$$\lim_{r\to\infty} \frac{\log_2 b_{2,d}(x_r)}{\log_2 x_r} = \lim_{r\to\infty} \log_2 \|A_0^{2r} A_1\|^{\frac{1}{2r+1}} = \log_2\, \rho(A_0),$$
$$\lim_{r\to\infty} \frac{\log_2 b_{2,d}(y_r)}{\log_2 y_r} = \lim_{r\to\infty} \log_2 \|(A_1 A_0)^r A_1\|^{\frac{1}{2r+1}} = \log_2 \sqrt{\rho(A_0 A_1)}\,.$$

We thus arrive at the following result.

Corollary 4. *For $k = 1, 2, 4$ the upper limit λ_2 and the lower limit λ_1 are attained at the sequences $\{y_n\}$ and $\{x_n\}$, respectively. For $k = 3, 5, 6$ the limits λ_2 and λ_1 are attained at the sequences $\{x_n\}$ and $\{y_n\}$, respectively.*

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Bibliography

[1] L. Euler, "Introductio in analysis infinitorum", Opera Omnia Series Prima Opera Math., vol. 8, Teubner, Leipzig 1922.

[2] B. Reznick, "Some binary partition functions", *Analytic number theory*, Urbana, IL 1989, Prog. Math., vol. 85, 1990, pp. 451–477.

[3] M. A. Stern, "Über eine zahlentheoretische Funktion", *J. für Math.* **55** (1858), 193–220.

[4] L. F. Klosinsky, G. L. Alexanderson, and A. P. Hillman, "The William Lowell Putman mathematical competition", *Amer. Math. Monthly* **91** (1984), 487–495.

[5] K. Mahler, "On a special functional equation", *J. London Math. Soc.* **15**:2 (1940), 115–123.

[6] N. G. de Bruijn, "On Mahler's partition problem", *Indag. Math.* (*N.S.*) **10** (1948), 210–220.

[7] D. E. Knuth, "An almost linear recurrence", *Fibonacci Quart.* **4** (1966), 117–128.

[8] G. C. Rota and G. Strang, "A note on the joint spectral radius", *Nederl. Akad. Wetensch. Verslag Afd. Natuurk.* **63** (1960), 379–381.

[9] J. C. Lagarias and Y. Wang, "The finiteness conjecture for the generalized spectral radius of a set of matrices", *Linear Algebra Appl.* **214** (1995), 17–42.

[10] V. Yu. Protasov, "Joint spectral radius and invariant sets of collections of linear operators", *Fund. Prikl. Mat.* **2** (1996), 205–271.

[11] M. A. Berger and Y. Wang, "Bounded semi-groups of matrices", *Linear Algebra Appl.* **166** (1992), 21–27.

[12] D. Collela and C. Heil, "Characterizations of scaling functions", *SIAM J. Matrix Anal. Appl.* **15** (1994), 1–23.

[13] T. Daubechies and J. C. Lagarias, "Two-scale difference equations. II. Local regularity, infinite products of matrices and fractals", *SIAM J. Math. Anal.* **23** (1992), 1031–1079.

[14] C. A. Micchelli and H. Prautzsch, "Uniform refinement of curves", *Linear Algebra Appl.* **114/115** (1989), 841–870.

[15] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, Springer-Verlag, New York 1999.

[16] R. F. Churchhouse, "Congruence properties of the binary partition function", *Proc. Camb. Phil. Soc.* **66** (1969), 371–376.
[17] A. Tanturri, "Sul numero delle partizioni d'un numero in potenze di 2", *Atti. Accad. Naz. Lincei* **27** (1918), 399–403.

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