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On the Asymptotics of the Binary Partition Function

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For an arbitrary integer $d \geq 2$, the binary partition function $b(k) = b(d,k)$ is defined on the set on nonnegative integers k as the total number of different binary expansions

$$k = \sum_{j=0}^{\infty} d_j 2^j,$$

where the "digits" d_j take values from the set $0, \dots, d-1$. For $d = \infty$, the quantity $b(\infty, k)$ is the number of such expansions with arbitrary nonnegative integer digits. Leonard Euler in [1] studied the partition function $b(\infty, k)$ in connection with certain power series. The asymptotic behavior of $b(\infty, k)$ as $k \to \infty$ was studied in various interpretations by K. Mahler, N. G. de Bruijn, D. E. Knuth, B. Reznick, and others (see [2] for numerous references). The first results for finite d were obtained by A. Tanturri in 1918 (see [3] and the two references in that work). Clearly, for $d = 2$, we have $b(k) \equiv 1$; for $d \geq 3$, such a binary expansion is not necessarily unique, and the following problem arises: characterize the asymptotic behavior of the function $b(k)$ as $k \to \infty$. B. Reznick in [2] showed that in the case $d = 2^{r+1}$, where $r \geq 0$ is an integer, one has $b(k) = C_r k^r + o(k^r)$ as $k \to \infty$. Here C_r is an effective constant. It was noted in [2] that this asymptotics can also be derived from results of A. Tanturri. For other even $d = 2n$, as was shown in [2], one has

$$C_n^1 k^{\log_2 n} \leq b(k) \leq C_n^2 k^{\log_2 n},$$

where C_n^1, C_n^2 are positive constants. Denote

$$\nu_1 = \liminf_{k\to\infty} k^{-\log_2 n} b(k), \qquad \nu_2 = \limsup_{k\to\infty} k^{-\log_2 n} b(k).$$

For any n, both ν_1 and ν_2 are positive and finite. If n is an integer power of two, then $\nu_1 = \nu_2$. So, in this case, $b(k) \sim c k^{\log_2 n}$ as $k \to \infty$. However, for a generic n, this is not always the case. In [2], B. Reznick showed (referring also to an earlier work of L. Carlitz [4]) that for $d = 6$, $n = 3$ we have $\nu_1 \neq \nu_2$. The question about the other n was formulated as an open problem. Does the property $\nu_1 = \nu_2$ hold only for the numbers n that are integer powers of 2? The following theorem gives the answer.

Theorem 1. *If $\nu_1 = \nu_2$, then $n = 2^r$ for some integer $r \geq 0$.*

In the proof of this theorem, we express ν_1 and ν_2 in terms of a special continuous function and show how to compute it approximately for any n (Proposition 2 and Remark 1).

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For odd values of d, the asymptotic behavior of $b(k)$ is more complicated; it was studied in [2] and [5]. Denote

$$p_1 = \liminf_{k\to\infty} \frac{\log b(k)}{\log k}, \qquad p_2 = \limsup_{k\to\infty} \frac{\log b(k)}{\log k}.$$

If d is even, then we always have $p_1 = p_2$, but for odd d this is not always the case. Already for $d = 3$ we have $p_1 < p_2$. Reznick in [2] computed these parameters explicitly for $d = 3$. In [5], they were computed for $d = 5, 7, 9, 11$ and 13. In all these cases, we have $p_1 < p_2$. Is this true for all odd d? In [2], it was shown that $p_1 \le \log_2(d/2) \le p_2$ and, moreover,

$$\limsup_{k\to\infty} k^{-\log_2(d/2)} b(k) = \infty.$$

In [5], it was proved that $p_1 = \log_2 \check{\rho}$ and $p_2 = \log_2 \hat{\rho}$, where

$$\check{\rho} = \lim_{s\to\infty} \min_{d_1,\dots,d_s\in\{0,1\}} \|T_{d_1}\cdots T_{d_s}\|^{1/s} \qquad \text{and} \qquad \hat{\rho} = \lim_{s\to\infty} \max_{d_1,\dots,d_s\in\{0,1\}} \|T_{d_1}\cdots T_{d_s}\|^{1/s}$$

are the so-called lower spectral radius and the joint spectral radius of the operators T_0, T_1. These operators act in $\mathbb{R}^{d-1}$ and are defined by their $(d-1)\times(d-1)$ matrices as follows: $(T_r)_{ij} = 1$ if $1 - r \le 2j - i \le d - r$, and $(T_r)_{ij} = 0$ otherwise ($r = 0, 1$). In [5], the following conjecture was stated (it is still unproved).

Conjecture 1. *If d is an odd integer, then*

$$\check{\rho} = \min\{\rho(T_0), \sqrt{\rho(T_0T_1)}\} \qquad \textit{and} \qquad \hat{\rho} = \max\{\rho(T_0), \sqrt{\rho(T_0T_1)}\},$$

where ρ denotes the (usual) spectral radius, i.e., the largest modulus of the eigenvalues.

In [5], this conjecture was proved for $d = 3, 5, \dots, 13$, which made it possible to compute explicitly the growth exponents p_1, p_2 for these values of d. Note than none of the results above (even if we assume Conjecture 1 to hold) implies that $p_1 < p_2$. In this paper, however, we establish the following.

Theorem 2. *For any odd d, $p_1 < \log_2(d/2) < p_2$.*

Let us start the proof of Theorems 1 and 2 by making some observations. Set $b(k) = 0$ for all integers $k < 0$. It can easily be checked that for every $k \in \mathbb{Z}$ we have the following recurrent relations:

$$b(2k) = \sum_{j=0}^{n-1} b(k-j), \qquad b(2k+1) = \sum_{j=0}^{n-1} b(k-j). \tag{1}$$

Denote $c_k = 2/d = 1/n$, $k = 0, \dots, 2n-1$, and $c_k = 0$ for all other k. Consider the so-called *subdivision operator* Γ, which acts on the space of bounded sequences ℓ_∞ by the formula

$$(\Gamma g)_k = \sum_i c_{k-2i} g_i,$$

where $g = (g_i)_{i\in\mathbb{Z}} \in \ell_\infty$. Now, take the initial sequence $g_0 = 1$, $g_i = 0$, $i \ne 0$. For every $j \ge 0$ we have

$$(\Gamma^j g)_k = \left(\frac{2}{d}\right)^j b(k) \qquad \text{for all} \quad k \le 2^j - 1. \tag{2}$$

This is easily shown by induction using (1). Now, we refer to the general theory of subdivision schemes (see, for instance, [6]). A subdivision scheme with positive coefficients always converges, provided

$$\sum_k c_{2k} = \sum_k c_{2k+1} = 1.$$

This means that $\|\varphi(2^{-j}\cdot) - \Gamma^j g\|_\infty \to 0$ as $j \to \infty$, where $\|\cdot\|_\infty$ is the uniform norm of the space ℓ_∞, φ is a unique continuous compactly supported solution of the *refinement equation*

$$\varphi\Big(\frac{x}{2}\Big) = \sum_{k=0}^{d-1} c_k \varphi(x-k) \tag{3}$$

such that $\int \varphi\, dt = 1$ (in our case all $c_k = 1/n$ for all k). Let us remark that $\operatorname{supp}\varphi \subset [0, d-1]$ (see [6]). Thus,

$$\delta_j = \max_{k \le 2^j - 1} |n^{-j} b(k) - \varphi(2^{-j}k)| \to 0 \qquad \text{as} \quad j \to \infty.$$

If we denote $\psi(x) = x^{-\log_2 n}\varphi(x)$, then we have the following assertion.

Proposition 1. *For every $j \ge 0$ and $k \le 2^j - 1$, the following inequality holds*:

$$\big|k^{-\log_2 n} b(k) - \psi(2^{-j}k)\big| \le (2^{-j}k)\delta_j, \tag{4}$$

where $\delta_j \to 0$ as $j \to \infty$.

Remark 1. For the case $d = 6$, $n = 3$, it was announced in [2] (without proof) that there exists a continuous function ψ possessing property (4). Now, we see that such a function does exist for every n and it is $\psi(x) = x^{-\log_2 n}\varphi(x)$, where φ is the continuous solution of the refinement equation (3) with $c_k = 2/d$. This solution can be found explicitly at all dyadic rational points $x = k/2^j$ (see [7]). Therefore, the limit

$$\lim_{j\to\infty} (2^j x)^{-\log_2 n} b(2^j x)$$

is also explicitly computed. Indeed, substituting $x = 2^{-j}k$ in (4), we see that this limit is equal to $\psi(x)$. Note also that for the rate of convergence of the subdivision scheme we have

$$\delta_j \le \|\varphi(2^{-j}\cdot) - \Gamma^j g\|_\infty \le \left(\frac{d-1}{d}\right)^j$$

(see [6]), therefore, we even know the rate of convergence of this limit.

Proposition 2. *Let $d \ge 2$ be an even integer, $b(k) = b(d,k)$ be the corresponding partition function. Then, for any integer $s \ge 1$,*

$$\nu_1 = \min_{x\in[2^{-s},2^{1-s}]} \psi(x), \qquad \nu_2 = \max_{x\in[2^{-s},2^{1-s}]} \psi(x).$$

Proof. Denote $M(s,j) = \{k/2^j,\ 2^{j-s} \le k \le 2^{j-s+1}\}$. Since φ is continuous, it follows that ψ is uniformly continuous on the segment $[2^{-s}, 2^{1-s}]$. Therefore,

$$\min_{x\in[2^{-s},2^{1-s}]} \psi(x) = \lim_{j\to+\infty} \min_{x\in M(s,j)} \psi(x).$$

By Proposition 1, the quantity

$$\min_{x\in M(s,j)} \psi(x)$$

is equivalent to

$$\min_{2^{j-s}\le k\le 2^{j-s+1}} k^{-\log_2 n} b(k)$$

as $j \to \infty$. Clearly,

$$\lim_{j\to\infty} \min_{2^{j-s}\le k\le 2^{j-s+1}} k^{-\log_2 n} b(k) = \liminf_{k\to\infty} k^{-\log_2 n} b(k).$$

Thus,

$$\inf_{x\in[2^{-s},2^{1-s}]} \psi(x) = \liminf_{k\to\infty} k^{-\log_2 n} b(k) = \nu_1.$$

The same holds for ν_2 with $\inf$ replaced by $\sup$. The proof is complete. □

Thus, we have found expressions for ν_1 and ν_2. These formulas make it possible to compute both ν_1 and ν_2 with arbitrary prescribed accuracy. To do this, one needs to compute the function φ approximately; this can be done, for instance, by the same subdivision schemes whose rate of convergence is known (Remark 1).

Corollary 1. *The following relations hold*:

$$\nu_1 = \inf_{x\in(0,1)} \psi(x), \qquad \nu_2 = \sup_{x\in(0,1)} \psi(x).$$

Proof of Theorem 1. If $\nu_1 = \nu_2$, then we see by Corollary 1 that on the interval $(0,1)$ we have $\varphi(x) \equiv Cx^{\log_2 n}$, where C is a constant. This implies that φ is an analytic function on each interval $(k, k+1)$, $k \in \mathbb{Z}$, Indeed, for $k \le 0$ this is proved. If this is true for all $k \le N$, then using (3) we obtain

$$\varphi(x) = n\varphi\biggl(\frac{x}{2}\biggr) - \sum_{k=0}^{2n-1} \varphi(x-k);$$

therefore, φ is analytic on $x \in (N, N+1)$. In the same way, we can show that for any integer $s \ge 1$ both one-sided limits $\varphi^{(s)}(k+0)$ and $\varphi^{(s)}(k-0)$ exist and are finite at all integer points k. Indeed, the left limit $\varphi^{(s)}(k-0)$ exists and finite for all $k \le 0$ (it is equal to zero). If it exists and is finite for all $k \le N-1$, then for $k = N$ we use the same equation, obtaining

$$\varphi^{(s)}(N-0)(x) = 2^{-s} n \varphi^{(s)}\biggl(\frac{N}{2} - 0\biggr) - \sum_{k=0}^{2n-1} \varphi^{(s)}(N-k-0)$$

(note that $N/2$ is either noninteger (so φ is analytic at that point) or is an integer smaller than N). Thus, the left limits exist and are finite at all integers. The right limits also exist since the function φ is symmetric. If $\log_2 n$ is not integer, then we reach a contradiction, because for $s > \log_2 n$ the limit $\varphi^{(s)}(+0) = (Cx^{\log_2 n})^{(s)}(+0)$ is infinite. The proof is complete. □

Proof of Theorem 2. Consider the refinement equation (3) again. It was shown in [6] that if all the coefficients c_k are nonnegative and $\sum_k c_k = 2$, then it possesses a unique, up to normalization, compactly supported solution φ in the space of distributions; this solution is a Borel probability measure, i.e., there exists a probability measure μ on $\mathbb{R}$ such that $(\varphi, f) = \int f\,d\mu$ for any test function f. This measure is supported on the segment $[0, d-1]$ and does not vanish identically on any interval in this segment. In [8], it was proved that such a measure is always continuous and is of pure type, i.e., either absolutely continuous ($\varphi \in L^1$) or purely singular. Moreover, if μ is absolutely continuous, then the polynomial

$$m(z) = \frac{1}{2}\sum_{k=0}^{d-1} c_k z^k$$

either vanishes at the point $z=-1$, or has a pair of symmetric roots, i.e., $m(z)=m(-z)=0$ for some complex $z\neq 0$. Now, for an odd $d=2n+1$, consider this refinement equation with $c_k=2/d$, $k=0,\dots,d-1$. We have

$$m(z)=\frac{1}{d}\sum_{k=0}^{d-1} z^k=\frac{z^d-1}{d(z-1)}.$$

Since $m(-1)=1/d\neq 0$ and m does not have symmetric roots, it follows that μ is purely singular. Let now Γ be the subdivision operator corresponding to this equation. Note that equality (2) holds for this operator as well. This is proved in the same way as in the case of even d by using (1), where the first sum is now taken from 0 to n, not to $n-1$. For any j, set

$$\varphi_j(x)=\sum_{k\in\mathbb{Z}}(\Gamma^j g)_k\,\chi(2^j x-k),$$

where χ is the characteristic function of the segment $[0,1]$. Then φ_j converges to φ in the sense of distributions as $j\to\infty$, i.e., for any test function f we have

$$(\varphi_j,f)\to(\varphi,f)=\int f\,d\mu$$

(for the proof see [6]). Now, we need the following fact proved in [5]: for any odd d there exist positive constants C_1,C_2 such that $C_1k^{p_1}\le b(k)\le C_2k^{p_2}$ for all $k\ge 1$. Using (2), we obtain

$$(\Gamma^j g)_k=\left(\frac{2}{d}\right)^j b(k)\,;$$

and, therefore,

$$C_1\left(\frac{2}{d}\right)^j k^{p_1}\le(\Gamma^j g)_k\le C_2\left(\frac{2}{d}\right)^j k^{p_2},$$

which for $k\in[2^{j-1},2^j)$ implies

$$C_1\frac{2}{d}\,k^{p_1-\log_2(d/2)}\le(\Gamma^j g)_k\le C_2k^{p_2-\log_2(d/2)}.$$

If $p_1-\log_2(d/2)\ge 0$, then $2C_1/d\le(\Gamma^j g)_k$ for all $k\in[2^{j-1},2^j)$. Hence, for every j, we have $\varphi_j(x)\ge 2C_1/d$ for all $x\in[1/2,1]$. Therefore, for the limit function φ we also have $\varphi\ge 2C_1/d$ on the segment $[1/2,1]$, which means that

$$\mu\ge\frac{2C_1}{d}\lambda,$$

where λ is the Lebesgue measure on $[1/2,1]$. This contradicts the singularity of μ. Thus,

$$p_1-\log_2\left(\frac{d}{2}\right)<0.$$

Now, assuming that

$$p_2-\log_2\left(\frac{d}{2}\right)\le 0,$$

we obtain $(\Gamma^j g)_k\le C_2$ for all $k\in[2^{j-1},2^j)$, and so $\varphi\le C_2$, which means $\mu\le C_2\lambda$ on the segment $[1/2,1]$. Since μ is not identically zero on this segment, this again contradicts the singularity of μ. Thus, $p_2>\log_2(d/2)$, which completes the proof. □

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