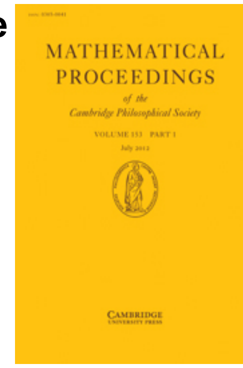

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Some arithmetical properties of *m*-ary partitions

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1. We denote by $t_m(n)$ the number of partitions of the positive integer n into non-decreasing parts which are positive or zero powers of a fixed integer $m > 1$ and we call $t_m(n)$ 'the m-ary partition function'. Mahler (1) obtained an asymptotic formula for $t_m(n)$, the first term of which is

$$\log t_m(n) \sim \frac{(\log n)^2}{2 \log m}.$$

Mahler's result was later improved by de Bruijn (2).

Following Churchhouse (3), we in particular denote the binary partition function $t_2(n)$ by $b(n)$. This function has been studied by Euler (4), Tanturri (5), (6), and recently by Churchhouse (3). With $m = 2$, equations (1), (2) and (3) below are due to Euler. Both Euler and Tanturri were concerned with deriving recurrence formulae for the precise calculation of $b(n)$, and Tanturri also found recurrence formulae involving the more general function $D(2^P, n)$ which denotes the number of partitions of n into powers of 2 of which 2^P is the maximum. However, Churchhouse seems to have been the first to discover that $b(n)$ has certain congruential periodicities, and he conjectured the property which we prove as our Theorem 1 below.

We also prove some further congruences and identities involving the m-ary partition functions, mainly when $m = p$, a prime. The main results are given in the four theorems below. Theorems 1 and 2 are concerned with the binary partition function $b(n)$. In Theorems 3 and 4 we give corresponding results in the case of the p-ary partition function for p an odd prime.

In the following we use $[a]$ to denote the integral part of a and $\binom{r}{s}$ to denote the binomial coefficient with the usual conventions. An empty sum is taken as zero.

THEOREM 1. *Let* $r > 0$ *and* $n \equiv 1 \pmod 2$. *Then*

$$b(2^{r+2}n) - b(2^r n) \equiv 2^{\mu(r)} \pmod{2^{\mu(r)+1}},$$

where $\mu(r) = [(3r+4)/2]$.

THEOREM 2. *Let* $r > 0$, *and put*

$$B(n) = b(4n) - b(n).$$

† The research described in this paper was done during the tenure of a research fellowship at the Atlas Computer Laboratory.

Then there exist integers $a_r(i)$ such that

$$B(2^r n) = \sum_{i=1}^{r-1} 2^{\nu(i)} a_r(i)\, B(2^{r-i} n) + 2^{(r+1)(r+2)/2} \sum_{k=1}^{n} \binom{r+k}{r+1} b(n-k),$$

where $\nu(i) = [(3i+1)/2]$.

Let $t_m(0) = 1$, and put $F_m(x) = \sum_{n=0}^{\infty} t_m(n) x^n \quad (|x| < 1)$.

Then
$$F_m(x) = \prod_{k=0}^{\infty} (1 - x^{m^k})^{-1}, \tag{1}$$

and it follows that $F_m(x)$ satisfies the functional equation

$$(1-x) F_m(x) = F_m(x^m), \tag{2}$$

so that
$$t_m(n) = t_m(n-1) + t_m\left(\frac{n}{m}\right), \tag{3}$$

where $t_m(n)$ is taken as zero if n is not a non-negative integer. If we further put

$$T_m(n) = t_m(mn) - t_m(n),$$

then
$$T_m(n) = t_m(mn - r) \quad (r = 1, \ldots, m), \tag{4}$$

and
$$\sum_{n=1}^{\infty} T_m(n) x^n = \frac{x}{1-x} F_m(x).$$

The two final theorems involving $t_m(n)$ we now state in terms of $T_m(n)$.

THEOREM 3. *Let $r > 0$ and p be an odd prime. Then*

$$T_p(p^r n) \equiv p^r A_{pr} \frac{n(n+1)}{2} T_p\left(\left[\frac{n}{p}\right] + 1\right) (\operatorname{mod} p^{r+1}),$$

where
$$A_{pr} = \left(\frac{p+1}{2}\right)^{r-1} \quad \textit{if} \quad p > 3,$$

$$A_{3r} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \quad \textit{if} \quad r \equiv \begin{Bmatrix} 1,\ 6,\ 7 \\ 2,\ 3,\ 5 \\ 0,\ 4, \end{Bmatrix} \quad (\operatorname{mod} 8).$$

THEOREM 4. *Let $r > 0$ and p be an odd prime. Then there exist integers $c_{pr}(i)$ such that*

$$T_p(p^r n) = \sum_{i=1}^{r-1} p^i c_{pr}(i)\, T_p(p^{r-i} n) + p^{r(r+1)/2} \sum_{k=0}^{n-1} \binom{n+k-1}{r-1} T_p(n-k).$$

The method we use below in proving the above results goes back to Ramanujan, and has been exploited since then by many writers, notably Watson (7). We use the technique of Atkin and O'Brien (8).

This paper is divided into five sections: in section 2 we introduce some notation. In section 3 we consider some properties of $F_m(x)$ and a related function. In section 4 we put $m = 2$ and prove Theorems 1 and 2, and in section 5 we put $m = p$, an odd prime, and prove Theorems 3 and 4.

2. We define a linear operator U_m acting on any power series $f(x) = \Sigma_{n \geqslant N} a(n) x^n$ by

$$U_m f(x) = \sum_{mn \geqq N} a(mn) x^n.$$

Clearly $$U_m(f_1(x) f_2(x^m)) = f_2(x)\, U_m f_1(x).$$

If ω is a primitive mth root of unity, it is easily seen that

$$U_m f(x) = \frac{1}{m} \sum_{\lambda=0}^{m-1} f(\omega^\lambda x^{1/m}). \tag{5}$$

For p prime we also define a valuation π_p by

$$p^{\pi_p(a)} | a, \quad p^{\pi_p(a)+1} \nmid a$$

for any integer a. If $a = 0$, we write conventionally $\pi_p(a) = \infty$ and regard any inequality $\pi_p(0) > b$ as valid. Clearly

$$\pi_p(ac) = \pi_p(a) + \pi_p(c),$$

$$\pi_p(a) \neq \pi_p(c) \quad \text{implies} \quad \pi_p(a+c) = \min(\pi_p(a), \pi_p(c)),$$

$$\pi_p(a) = \pi_p(c) \quad \text{implies} \quad \pi_p(a+c) \geqslant \pi_p(a),$$

$$\pi_2(a) = \pi_2(c) \quad \text{implies} \quad \pi_2(a+c) > \pi_2(a).\dagger$$

3. We now write $$g(x) = \frac{1}{1-x}.$$

It is clear that all the roots of the equation

$$\left(1 - \frac{1}{y}\right)^m = x, \tag{6}$$

regarded as an equation in y, are given by

$$y = g(\omega^\lambda x^{1/m}) \quad (0 \leqslant \lambda < m),$$

where ω is a primitive mth root of unity. Thus, if S_r denotes the sum of the rth powers of the roots of (6), we have, by (5),

$$U_m g^r(x) = \frac{1}{m} S_r.$$

Writing (6) as $$y^m + g(x) \sum_{k=1}^{m} (-1)^k \binom{m}{k} y^{m-k} = 0,$$

we find by Newton's formulae, writing $g = g(x)$, that

$$S_r = \sum_{k=1}^{r-1} (-1)^{k+1} \binom{m}{k} g S_{r-k} + r(-1)^{r+1} \binom{m}{r} g. \tag{7}$$

Now, let $$h_r = h_r(x) = g^r - g^{r-1} \quad (r \geqslant 1),$$

$$V_{mr} = U_m h_r.$$

† It is because of this feature, peculiar to $p = 2$, that we in Theorem 1 are able to give the *exact* power of 2 dividing the expression involved.

Then
$$V_{mr} = \frac{1}{m}(S_r - S_{r-1}),$$
and (7) gives
$$V_{mr} = \sum_{k=0}^{r-2} (-1)^k \binom{m}{k+1} g V_{m,r-k-1} \quad (r \geqslant 2), \tag{8}$$
since $V_{m1} = h_1$.

By induction on r, it is now easily proved that
$$V_{mr} = \sum_{i=0}^{r-2} \alpha_{mr}(i)\, h_{i+2} \quad (r \geqslant 2), \tag{9}$$
and
$$\left.\begin{aligned} \alpha_{mr}(i) &= \sum_{k\geqslant 0} (-1)^k \binom{m}{k+1} \alpha_{m,r-k-1}(i-1) \quad (1 \leqslant i \leqslant r-2), \\ \alpha_{mr}(0) &= (-1)^r \binom{m}{r-1}, \end{aligned}\right\} \tag{10}$$
where in fact $\alpha_{mr}(i) = 0$ if $(i+1)m < r-1$ (i.e. if $i < [(r-2)/m]$).

We now prove

LEMMA 1. *For p prime, we have*
$$\pi_p(\alpha_{pr}(i)) \geqslant i+1-\left[\frac{r-i-2}{p-1}\right] \quad (r \geqslant 2).$$

Proof. We use induction on r. From (10) we see that Lemma 1 holds for $r = 2$. Assuming Lemma 1 for all r, $2 \leqslant r < R$, for some $R > 2$, we obtain from (10)
$$\pi_p(\alpha_{pR}(i)) \geqslant \min_{k\geqslant 0} \left\{\pi_p\left(\binom{p}{k+1}\right) + i - \left[\frac{R-k-i-2}{p-1}\right]\right\} = i+1-\left[\frac{R-i-2}{p-1}\right],$$
looking separately at the cases $k =$ and $\neq p-1$. This is Lemma 1 for $r = R$, completing the proof.

Now, let
$$H_{mr} = F_m^{-1}(x) \sum_{n=1}^{\infty} T_m(m^r n)\, x^n \quad (r \geqslant 0).$$
Then $H_{m0} = h_1$, and
$$H_{mr} = U_m(g H_{m,r-1}) \quad (r > 0).$$
Especially we find $H_{m1} = m h_2$, which shows that
$$T_m(mn) \equiv 0 \pmod{m}.$$

By induction on r it is now easily shown that
$$H_{mr} = \sum_{i=0}^{r-1} \beta_{mr}(i)\, h_{i+2} \quad (r > 0), \tag{11}$$
where all the $\beta_{mr}(i)$ are integers, and
$$\beta_{mr}(i) = \sum_{j\geqslant \max(0,i-1)} \alpha_{m,j+3}(i)\, \beta_{m\, r-1}(j) \quad (r > 1). \tag{12}$$
Especially we see that $\beta_{mr}(r-1) = m^{r(r+1)/2}$.

4. In this section we put $m = 2$ and drop the suffix m in our notation. From (10) it follows that

$$\alpha_r(i) = (-1)^{r-i}\, 2^{2i-r+3}\binom{i+1}{r-i-2}.$$

Let
$$K_r = H_r + H_{r-1} \quad (r \geqslant 1).$$

Then
$$K_r = F^{-1}(x) \sum_{n=1}^{\infty} (b(2^{r+1}n) - b(2^{r-1}n))\, x^n. \tag{13}$$

By (11) we get
$$K_2 = 2^3 h_3. \tag{14}$$

Similarly to (11) we find
$$K_r = \sum_{i=1}^{r-1} \gamma_r(i)\, h_{i+2} \quad (r \geqslant 2),$$

where
$$\gamma_r(i) = \sum_{j \geqslant \max(1,\, i-1)} a_{j+3}(i)\, \gamma_{r-1}(j) \quad (j \geqslant 3). \tag{15}$$

For integral r we define a symbol $*_r$ to mean $=$ if r is odd and $>$ if r is even. Now we have

LEMMA 2. *If* $r \geqslant 2$, *then*
$$\pi(\gamma_r(1)) = \left[\frac{3r+1}{2}\right],$$
$$\pi(\gamma_r(2)) *_r \left[\frac{3r+4}{2}\right],$$
$$\pi(\gamma_r(i)) \geqslant \left[\frac{3r+i^2}{2}\right] \quad (i > 2).$$

Proof. From (14) we see that Lemma 2 holds for $r = 2$. Assuming Lemma 2 for all r, $2 \leqslant r < R$, for some $R > 2$, we obtain from (15)

$$\pi(\gamma_R(i)) \geqslant \min_{j \geqslant i-1}\left(2i - j + \left[\frac{3R-3+j^2}{2}\right]\right),$$
$$= \left[\frac{3R+i^2}{2}\right] \quad \text{if} \quad (i \geqslant 2).$$

Further we have
$$\gamma_R(1) = -2^2\gamma_{R-1}(1) + \gamma_{R-1}(2).$$

Now,
$$2 + \pi(\gamma_{R-1}(1)) *_R \left[\frac{3R+1}{2}\right],$$
$$\pi(\gamma_{R-1}(2)) *_{R-1} \left[\frac{3R+1}{2}\right];$$

thus
$$\pi(\gamma_R(1)) = \left[\frac{3R+1}{2}\right].$$

Similarly we find that
$$\pi(\gamma_R(2)) *_R \left[\frac{3R+4}{2}\right].$$

This completes the proof of Lemma 2.

By (13) and Lemma 2 we now have if $r > 0$,

$$b(2^{r+2}n) - b(2^r n) \equiv 2^{\mu(r)} d(n) \pmod{2^{\mu(r)+1}},$$

where $\mu(r)$ is given as in Theorem 1, and

$$\sum_{n=1}^{\infty} d(n) x^n = h_3(x) F(x).$$

Thus
$$d(n) = \sum_{m=1}^{n} \frac{m(m+1)}{2} b(n-m)$$
$$\equiv n \pmod 2,$$

since $b(n) \equiv 0 \pmod 2$ if $n \geqslant 2$. This completes the proof of Theorem 1.

Now, $\gamma_r(r-1) = \beta_r(r-1)$, and

$$K_r = 2^{r(r+1)/2} h_{r+1} + \sum_{i=1}^{r-2} 2^{[(3r+i^2)/2]} \delta_r(i) h_{i+2},$$

where all the $\delta_r(i)$ are integers by Lemma 2. For fixed r there certainly exist constants $Z_j = Z_j(r)$ such that

$$K_r - 2^{r(r+1)/2} h_{r+1} = \sum_{j=2}^{r-1} Z_j K_j,$$

and the Z_j are given as the solution of the linear equations

$$Z_{r-k} + \sum_{i=1}^{k-1} 2^{[(3i+1)/2]} \delta_{r-k+i}(r-k-1) Z_{r-k+i} = 2^{[(3k+1)/2]} \delta_r(r-k-1) \quad (k = 1, \ldots, r-1).$$

From this we see that

$$Z_{r-k} = 2^{[(3k+1)/2]} a_{r-1}(k) \quad (k = 1, \ldots, r-1),$$

where all the $a_r(k)$ are integers. Thus we have

$$B(2^r n) = \sum_{i=1}^{r-1} 2^{[(3i+1)/2]} a_r(i) B(2^{r-i} n) + 2^{(r+1)(r+2)/2} e(n),$$

where
$$\sum_{n=1}^{\infty} e(n) x^n = h_{r+2}(x) F(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{r+k}{r+1} b(n-k) x^n.$$

This completes the proof of Theorem 2.

5. In this section we take $m = p$, an odd prime, and when no ambiguity is likely to arise, we drop the suffix m in the notations of sections 1, 2 and 3.

LEMMA 3. $\pi(\beta_r(i)) \geqslant r + \dfrac{i(i+1)}{2}$ $(r > 0)$.

This follows immediately from (12), Lemma 1, and induction on r. Now we have

$$H_r = \sum_{i=0}^{r-1} p^{r+i(i+1)/2} \epsilon_r(i) h_{i+2} \quad (r > 0),$$

where all the $\epsilon_r(i) = \epsilon_{pr}(i)$ are integers. From (10) and (12) we get

$$\epsilon_{pr}(0) \equiv -\frac{p-1}{2} \epsilon_{p,r-1}(0) \pmod p \quad \text{if} \quad p > 3,$$

$$\epsilon_{3r}(0) \equiv -\epsilon_{3,r-1}(0) + \epsilon_{3,r-2}(0) \pmod 3.$$

Now $\epsilon_{p1}(0) = 1$, $\epsilon_{32}(0) = -1$, and we find that

$$\epsilon_r(0) \equiv A_r \pmod{p},$$

where $A_r = A_{pr}$ is given as in Theorem 3.

Now,
$$T(p^r n) \equiv p^r A_r f(n) \pmod{p^{r+1}},$$
where
$$\sum_{n=1}^{\infty} f(n)\, x^n = h_2(x)\, F(x).$$
Thus
$$f(n) = \sum_{k=0}^{n-1} (n-k)\, t(k).$$

By means of the relations (3) and (4), it is easily shown that

$$f(n) \equiv \frac{n(n+1)}{2} T\left(\left[\frac{n}{p}\right]+1\right) \pmod{p}.$$

This completes the proof of Theorem 3.

By a similar technique to that which we applied at the end of section 4 we further get

$$H_r - p^{r(r+1)/2} h_{r+1} = \sum_{i=1}^{r-1} p^i c_r(i)\, H_{r-i},$$

where all the $c_r(i)$ are integers. Thus we have

$$T(p^r n) = \sum_{i=1}^{r-1} p^i c_r(i)\, T(p^{r-i} n) + p^{r(r+1)/2} g(n),$$
where
$$\sum_{n=1}^{\infty} g(n)\, x^n = h_{r+1}(x)\, F(x) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{r+k-1}{r-1} T(n-k)\, x^n.$$

This completes the proof of Theorem 4.

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