STATISTICAL PROPERTIES OF THE CALKIN–WILF TREE: REAL AN p–ADIC DISTRIBUTION

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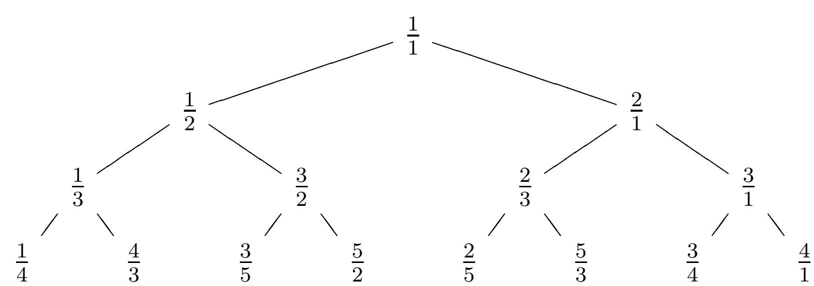
Abstract. We examine statistical properties of the Calkin–Wilf tree and give number-theoretical applications.

1. A mean-value related to the Calkin–Wilf tree

The Calkin–Wilf tree is generated by the iteration

$$\frac{a}{b} \quad \mapsto \quad \frac{a}{a+b}\,, \quad \frac{a+b}{b},$$

starting from the root $\frac{1}{1}$; the number $\frac{a}{a+b}$ is called the left child of $\frac{a}{b}$ and $\frac{a+b}{b}$ the right child; we also say that $\frac{a}{b}$ is the mother of its children. Recently, Calkin & Wilf [1] have shown that this tree contains any positive rational number once and only once, each of which represented as a reduced fraction. The first iterations lead to



Reading the tree line by line, the Calkin–Wilf enumeration of $\mathbb{Q}^+$ starts with

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \ldots$$

As recently pointed out by Reznick [10], this sequence was already investigated by Stern [12] in 1858. This sequence satisfies also the iteration

$$x_1 = 1, \quad x_{n+1} = 1/(2[x_n] + 1 - x_n),$$

where $[x]$ denotes the largest integer $\leq x$; this observation is due to Newman (cf. [8]), answering a question of D.E. Knuth, resp. Vandervelde & Zagier (cf. [11]).

The Calkin–Wilf enumeration of the positive rationals has many interesting features. For instance, it encodes the hyperbinary representations of all positive integers (see [1]). Furthermore, it can be used as model for the game *Euclid* first formulated by Cole & Davie [2]; see Hofmann, Schuster & Steuding [5]. In this short note we are concerned with statistical properties of the Calkin–Wilf tree.

We write the nth generation of the Calkin–Wilf tree as $\mathcal{CW}^{(n)} = \{x_j^{(n)}\}_j$, where the $x_n^{(j)}$ are the elements ordered according to their appearance in the nth line of the Calkin–Wilf tree. So $\mathbb{Q}^+ = \bigcup_{n=1}^{\infty} \mathcal{CW}^{(n)}$. Obviously, $\mathcal{CW}^{(n)}$ consists of 2^{n-1} elements. Denote by $\Sigma(n)$ the sum of all elements of the nth generation of the Calkin–Wilf tree,

$$\Sigma(n) = \sum_{j=1}^{2^{n-1}} x_j^{(n)}.$$

Our first result gives the mean-value of the elements of the nth generation of the Calkin–Wilf tree:

Theorem 1. *For any $n \in \mathbb{N}$,*

$$\Sigma(n) = 3 \cdot 2^{n-2} - \frac{1}{2}.$$

This result may be interpreted as follows. We observe that $x_1^{(n)} = \frac{1}{n}$ and $x_{2^{n-1}}^{(n)} = \frac{n}{1}$ for all $n \in \mathbb{N}$, and thus $\mathcal{CW}^{(n)}$ is supported on an unbounded set as $n \to \infty$. However, the average value of the 2^{n-1} elements of the nth generation $\mathcal{CW}^{(n)}$ is approximately $\frac{3}{2}$, which is, surprisingly, a finite number. This has a simple explanation: in some sense, *small* values are taken in earlier generations than *large* values. For instance, in each generation $\mathcal{CW}^{(n)}$ takes as many values form the interval $(0,1)$ as from $(1,\infty)$. This result was also recently proved by Reznick [10]; his proof differs slightly from our argument.[1]

[1]The problem of determining the average value of the Calkin–Wilf tree was posed by the second named author as a problem in the problem session of the IV International conference on analytic and probabilistic number theory in Palanga 2006; an independent solution was given by Eduard Wirsing.

Proof by induction on n. The statement of the theorem is correct for $n = 1$ and $n = 2$. Now suppose that $n \geq 3$. In order to prove the statement for n we first observe a symmetry in the Calkin–Wilf tree with respect to its middle: for $n \geq 2$,

$$x_j^{(n)} = \frac{a}{b} \qquad \Longleftrightarrow \qquad x_{2^{n-1}+1-j}^{(n)} = \frac{b}{a}\ ; \tag{1}$$

this is easily proved by another induction on n (and we leave its simple verification to the reader). Further, we note that $x_j^{(n)} \leq 1$ if and only if j is odd; here equality holds if and only if $n = 1$.

Now we start to evaluate $\Sigma(n)$. For this purpose we compute

$$y_j^{(n)} := \begin{cases} x_j^{(n)} + x_{2^{n-1}-j}^{(n)} & \text{for } j = 1, 2, \dots, 2^{n-2} - 1, \\ x_j^{(n)} + x_{2j}^{(n)} & \text{for } j = 2^{n-2}, \end{cases}$$

and add these values over $j = 1, 2, \dots, 2^{n-2}$. Clearly, $\Sigma(n) = \sum_{j=1}^{2^{n-2}} y_j^{(n)}$.

First, assume that j is odd. Then both, $x_j^{(n)}$ and $x_{2^{n-1}-j}^{(n)}$ are strictly less than 1. In view of (1) the mothers of $x_j^{(n)}$ and $x_{2^{n-1}-j}^{(n)}$ are of the form $\frac{a}{b}$ and $\frac{b}{a}$, respectively. Hence,

$$x_j^{(n)} + x_{2^{n-1}-j}^{(n)} = \frac{a}{a+b} + \frac{b}{a+b}$$

and thus we find $y_j^{(n)} = 1$ in this case.

Next, we consider the case that j is even. Then both, $x_j^{(n)}$ and $x_{2^{n-1}-j}^{(n)}$ are strictly greater than 1. If the mothers of $x_j^{(n)}$ and $x_{2^{n-1}-j}^{(n)}$ are of the form $\frac{a}{b}$ and $\frac{a'}{b'}$, respectively, then

$$x_j^{(n)} = \frac{a+b}{b} = 1 + \frac{a}{b} \qquad \text{and} \qquad x_{2^{n-1}-j}^{(n)} = 1 + \frac{a'}{b'}.$$

Hence, we find for their sum

$$x_j^{(n)} + x_{2^{n-1}-j}^{(n)} = 2 + \frac{a}{b} + \frac{a'}{b'}$$

and so $y_j^{(n)} = 2 + y_k^{(n-1)}$, where $y_k^{(n-1)}$ is either the sum of two elements $x_k^{(n-1)}$ and $x_{2^{n-2}-k}^{(n-1)}$ or the sum of $x_{2^{n-3}}^{(n-1)}$ and $x_{2^{n-2}}^{(n-1)}$.

It remains to combine both evaluations. Since both cases appear equally often, namely each 2^{n-3} times, we obtain the recurrence formula

$$\Sigma(n) = \Sigma(n-1) + (1+2) \cdot 2^{n-3},$$

being valid for $n \geq 3$. This implies the assertion of the theorem. ■

2. An application to finite continued fractions

Theorem 1 has a nice number-theoretical interpretation. It is well-known that each positive rational number x has a representation as a finite (regular) continued fraction

$$x = a_0 + \cfrac{1}{a_1 + \ddots + \cfrac{1}{a_{m-1} + \frac{1}{a_m}}}$$

with $a_0 \in \mathbb{N} \cup \{0\}$ and $a_j \in \mathbb{N}$ for some $m \in \mathbb{N} \cup \{0\}$. In order to have a unique representation, we assume that $a_m \geq 2$ if $m \in \mathbb{N}$. We shall use the standard notation $x = [a_0, a_1, \ldots, a_m]$. Continued fractions are of special interest in the theory of diophantine approximation.

As Bird, Gibbons & Lester [3] showed, the nth generation of the Calkin–Wilf tree consists exactly of those rationals having a continued fraction expansion $[a_0, a_1, \ldots, a_m]$ for which the sum of the partial quotients a_j is constant n, the continued fractions of even length in the left subtree, and the continued fractions with odd length in the right subtree. Thus Theorem 1 yields

Corollary 2. *For any $n \in \mathbb{N}$,*

$$2^{1-n} \sum_{a_0+a_1+\ldots+a_m=n} [a_0, a_1, \ldots, a_m] = \frac{3}{2} - 2^{-n}.$$

One can use the approach via continued fractions to locate any positive rational in the tree. This observation is due to Bird, Gibbons & Lester [3] (actually, their reasoning is based on Graham, Knuth & Patasnik [4] who gave such a description for the related Stern–Brocot tree). Given a reduced fraction x in the Calkin–Wilf tree with continued fraction expansion

$$x = [a_0, a_1, \ldots, a_{m-2}, a_{m-1}, a_m],$$

we associate the path

$$\begin{aligned} &\mathsf{L}^{a_m - 1}\mathsf{R}^{a_{m-1}}\mathsf{L}^{a_{m-2}} \cdots \mathsf{L}^{a_1}\mathsf{R}^{a_0} \quad \text{if} \quad m \text{ is odd, and} \\ &\mathsf{R}^{a_m - 1}\mathsf{L}^{a_{m-1}}\mathsf{R}^{a_{m-2}} \cdots \mathsf{R}^{a_1}\mathsf{L}^{a_0} \quad \text{if} \quad m \text{ is even;} \end{aligned}$$

note that $a_m - 1 \geq 1$ for $m \in \mathbb{N}$. The notation R^a with $a \in \mathbb{N} \cup \{0\}$ means: a steps to the right, whereas L^b with $b \in \mathbb{N} \cup \{0\}$ stands for b steps to the left. Then, starting from the root $\frac{1}{1}$ and following this path from left to right, we end up with the element x. This follows easily from the iteration with which the tree was build; notice that this claim is essentially already contained in Lehmer [7] (this was also observed by Reznick [10]).

Corollary 3. *Given any non-empty interval (α, β) in $\mathbb{R}^+$, and any finite path in the Calkin–Wilf tree, there exists a continuation of this path which contains a rational number from the interval (α, β).*

Proof. We expand α and β into continued fractions, $\alpha = [a_0, a_1, \ldots]$ and $\beta = [b_0, b_1, \ldots]$, say. Let k be the least index such that $a_k \neq b_k$. According to the parity of k we have $a_k < b_k$ (if k is even) or $a_k > b_k$ (if k is odd). Without loss of generality we may assume that $|b_k - a_k| \geq 2$ (since otherwise we may consider a subinterval of (α, β)). Moreover we may suppose that the path in question is starting from the root and is given in the form $\mathsf{L}^{c_m - 1}\mathsf{R}^{c_{m-1}}\mathsf{L}^{c_{m-2}} \cdots \mathsf{L}^{c_1}\mathsf{R}^{c_0}$ (the other case may be treated analogously). Then we construct a rational number x by assigning the finite continued fraction

$$x = [a_0, a_1, \ldots, a_{k-1}, x_k, x_{k+1}, c_0, c_1, \ldots, c_{m-2}, c_{m-1}, c_m],$$

where $x_k := \min\{a_k, b_k\} + 1$ and x_{k+1} denotes the string 1 if k is odd, resp. $1, 1$ if k is even. Since $b_j = a_j$ for $0 \leq j < k$ and

$$\min\{a_k, b_k\} < x_k < \max\{a_k, b_k\},$$

it follows that $\alpha < x < \beta$. Since the length of the continued fraction expansion has the same parity as m (thanks to the definition of x_{k+1}), the element x can be reached by the path $\mathsf{L}^{c_m - 1}\mathsf{R}^{c_{m-1}}\mathsf{L}^{c_{m-2}} \cdots \mathsf{L}^{c_1}\mathsf{R}^{c_0}$. This proves the corollary. ∎

3. A random walk on the Calkin–Wilf tree

Starting with $X_1 = \frac{1}{1}$, we define a sequence of random variables by the following iteration: if $X_n = \frac{a}{b}$, then $X_{n+1} = \frac{a}{a+b}$ with probability $\frac{1}{2}$ and $X_{n+1} = \frac{a+b}{b}$ with probability $\frac{1}{2}$. The sequence $\{X_n\}$ may be regarded as a random walk on the Calkin–Wilf tree where n is a discrete time parameter.

Theorem 4. *Let (α, β) be any non-empty interval in $\mathbb{R}^+$. Then, with probabilty 1, the random walk $\{X_n\}$ visits the interval (α, β), i.e., with probability 1, there exists $m \in \mathbb{N}$ such that $x_m \in (\alpha, \beta)$.*

Proof. The interval (α, β) contains a non-empty subinterval $[A, B]$ such that for any $\zeta \in [A, B]$ the initial partial quotients $c_0, c_1, \ldots, c_m$ are identical: $\zeta = [c_0, c_1, \ldots, c_m, \ldots]$. Hence, with the interval $[A, B]$ we may associate a path pattern $\mathsf{L}^{c_m - 1}\mathsf{R}^{c_{m-1}}\mathsf{L}^{c_{m-2}} \cdots \mathsf{L}^{c_1}\mathsf{R}^{c_0}$ in the Calkin-Wilf tree such that any path in the tree starting from the root and ending with $\mathsf{L}^{c_m - 1}\mathsf{R}^{c_{m-1}}\mathsf{L}^{c_{m-2}} \cdots \mathsf{L}^{c_1}\mathsf{R}^{c_0}$ points to an element in $[A, B]$. Since the probability is $\frac{1}{2}$ for both $\frac{a}{b} \mapsto \frac{a}{a+b}$ and $\frac{a}{b} \mapsto \frac{a+b}{b}$, each pattern of fixed length m appears with the same probability and so we may restrict on the path pattern R^k.

In the case $k = 1$ we find in each generation exactly one which ends with R but does not contain any R before (actually, this is $L^{n-1}R$ in generation n). Adding up all probabilities for these paths, we get

$$\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1,$$

and so, with probability 1, the random walk X_n will go to the right child for some n. Now assume that the statement is true for k. We shall show that then it is also true for $k+1$. For each path of the form XR^k, where X is any combination of powers of L and R, there are two paths XR^kL and XR^{k+1}, so by induction the probability that the random walk eventually follows the path R^{k+1} is at least $\frac{1}{2}$. However, for each path XR^k one also has to consider the subtrees starting from XR^kL^d for $d = 1, 2, \ldots$, each of which containing paths which end R^{k+1}. By self-similarity, the probability that the random walk eventually follows the path R^{k+1} is

$$\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j \sum_{d=1}^{\infty} \left(\frac{1}{2}\right)^d = 1.$$

This proves the theorem. ■

4. Statistical properties of the Calkin–Wilf tree

In view of Corollary 2 it is interesting to have a better understanding of the statistics of the Calkin–Wilf tree. The following theorem gives the limit distribution function in explicit form.

Theorem 5. *Let $F_n(x)$ denote the distribution function of the $n-th$ generation, i.e.,*

$$F_n(x) = 2^{1-n}\#\{j \,:\, x_j^{(n)} \le x\}.$$

Then uniformly $F_n(x) \to F(x)$, where

$$F([a_0, a_1, a_2, a_3, ...]) = 1 - 2^{-a_0} + 2^{-(a_0+a_1)} - 2^{-(a_0+a_1+a_2)} + ...$$

(for rational numbers $x = [a_0, a_1, \ldots]$ this series terminates at the last non-zero partial quotient of the continued fraction). Thus, $F(0) = 0$, $F(\infty) = 1$, and $F(x)$ is a monotonically increasing function. Moreover, $F(x)$ is continuous and singular, i.e., $F'(x) = 0$ almost everywhere.

Proof. Let $x \ge 1$. One half of the fractions in the $n+1-$st generation do not exceed 1, and hence also do not exceed x. Further,

$$\frac{a+b}{b} \le x \quad \Longleftrightarrow \quad \frac{a}{b} \le x - 1.$$

Hence,

$$2F_{n+1}(x) = F_n(x-1) + 1, \quad n \ge 1.$$

Now assume $0 < x < 1$. Then

$$\frac{a}{a+b} \le x \quad \Longleftrightarrow \quad \frac{a}{b} \le \frac{x}{1-x}.$$

Therefore,

$$2F_{n+1}(x) = F_n\Big(\frac{x}{1-x}\Big).$$

The distribution function F, defined in the formulation of the theorem, satisfies the functional equation

$$2F(x) = \begin{cases} F(x-1) + 1 & \text{if} \quad x \ge 1, \\ F(\frac{x}{1-x}) & \text{if} \quad 0 < x < 1. \end{cases}$$

For instance, the second identity is equivalent to $2F(\frac{t}{t+1}) = F(t)$ for all positive t. If $t = [b_0, b_1, ...]$, then $\frac{t}{t+1} = [0, 1, b_0, b_1, ..]$ for $t \ge 1$, and $\frac{t}{t+1} = [0, b_1 + 1, b_2, ...]$ for $t < 1$, and the statement follows immediately.

Now define $\delta_n(x) = F(x) - F_n(x)$. In order to prove the first assertion of the theorem, the uniform convergence $F_n \to F$, it is sufficient to show that

$$\sup_{x \geq 0} |\delta_n(x)| \leq 2^{-n}. \tag{2}$$

It is easy to see that the assertion is true for $n = 1$. Now suppose the estimate is true for n. In view of the functional equation for both $F_n(x)$ and $F(x)$, we have

$$2\delta_{n+1}(x) = \delta_n\Big(\frac{x}{1-x}\Big)$$

for $0 < x < 1$, which gives $\sup_{0 \leq x < 1} |\delta_{n+1}(x)| \leq 2^{-n-1}$. Moreover, we have

$$2\delta_{n+1}(x) = \delta_n(x-1)$$

for $x \geq 1$, which yields the same bound for $\delta_n(x)$ in the range $x \geq 1$. This proves (2).

Clearly, F, as a distribution function, is monotonic; obviously, it is also continuous. It remains to prove that $F(x)$ is singular. Given an irrational number $\alpha = [a_0, a_1, a_2, ...]$, we consider the sequence

$$\alpha_n = [a_0, a_1, ..., a_{n-1}, a_n + 1, a_{n+1}, ...];$$

obviously, α_n is the real number which is defined by the continued fraction expansion of α, where the nth partial quotient a_n is replaced by $a_n + 1$. Depending on the parity of n, α_n is less than or greater than α. Thus, any real number y, which is sufficiently close to α, is contained between two terms of the sequence, α_L and α_{L+2} say. Then

$$\Big|\frac{F(y) - F(\alpha)}{y - \alpha}\Big| \leq \Big|\frac{F(\alpha_L) - F(\alpha)}{\alpha_{L+2} - \alpha}\Big|.$$

From the explicit form of F we deduce

$$|F(\alpha_L) - F(\alpha)| \leq \frac{1}{2} 2^{-(a_0 + a_1 + ... + a_L)}.$$

On the other hand,

$$\begin{aligned} |\alpha_{L+2} - \alpha| &\geq ([a_1, a_2, ..., a_{L+2} + 1, ...] - [a_1, a_2, ..., a_{L+2}, ...])(a_0 + 1)^{-2} \\ &\geq \Big((a_0 + 1)(a_1 + 1)...(a_{L+2} + 1)\Big)^{-2} \end{aligned}$$

by induction. Thus,

$$\left|\frac{F(y)-F(\alpha)}{y-\alpha}\right| \leq 2^{1-(a_0+a_1+\ldots+a_L)} \prod_{i=1}^{L+2} (a_i+1)^2.$$

The theorem of Khinchin ([9], p. 86, implies that $\prod_{i=1}^{n}(a_i+1)^{1/n}$ tends to a fixed constant limit almost everywhere. On the other hand, the same reasoning shows that $\frac{1}{n}\sum_{i=1}^{n} a_n$ tends to infinity for almost all x. Thus, almost everywhere the limit

$$\lim_{y\to\alpha}(F(y)-F(\alpha))(y-\alpha)^{-1}$$

exists and is equal 0. This finishes the proof of the theorem. ∎

By the same argument as for the singular behaviour of F we can show that $F'(\frac{\sqrt{5}+1}{2}) = \infty$. Actually, the terms of $\mathcal{CW}^{(n)}$ are densely concentrated around numbers with $F'(x) = \infty$ and scarcely around those where $F'(x) = 0$. The value of $F(x)$ is rational iff x is either rational or quadratic irrationality, e.g.

$$F(1) = \frac{1}{2}, \quad F(\sqrt{2}) = \frac{3}{5}, \quad F((\sqrt{5}+1)/2) = \frac{2}{3}.$$

This follows immediately from Lagrange's theorem which characterizes the quadratic irrationals by their eventually periodic continued fraction expansion. For Euler's number $e = [2, \overline{1, 2n, 1}]$ we find that $F(e)$ can be expressed in terms of special values of Jacobi theta functions.

5. Characteristics of the distribution function

In view of Corollary 2, the mean of the distribution function F is $\frac{3}{2}$. Since F has a tail of exponential decay, more precisely $1 - F(x) = O(2^{-x})$, it follows that all moments exist. For $k \in \mathbb{N}_0$, the kth moment is defined by

$$M_k = \int_0^\infty x^k \, \mathrm{d}F(x).$$

In order to give an asymptotic formula for M_k let

$$m_k = \int_0^\infty \Big(\frac{x}{x+1}\Big)^k \, \mathrm{d}F(x)$$

We will see that the generating function of m_k has some interesting properties. Let $\omega(x)$ be a continuous function of at most power growth: $\omega(x) \ll x^T$ as $x \to \infty$. By the functional equation for F we find $F(x+n) = 1 - 2^{-n} + 2^{-n}F(x)$, $x \geq 0$. Hence

$$\begin{aligned} \int_0^\infty \omega(x)\,\mathrm{d}F(x) &= \sum_{n=0}^\infty \int_0^1 \omega(x+n)\,\mathrm{d}F(x+n) \\ &= \int_0^1 \sum_{n=0}^\infty \frac{\omega(x+n)}{2^n}\,\mathrm{d}F(x); \end{aligned}$$

these integrals exist in view of our assumptions and the fact that $F(x)$ has a tail of exponential decay. Let $x = \frac{t}{t+1}$ for $t \geq 0$. Since $F(\frac{t}{t+1}) = \frac{1}{2}F(t)$, this change of variables gives

$$\int_0^\infty \omega(x)\,\mathrm{d}F(x) = \sum_{n=0}^\infty \int_0^\infty \frac{\omega(\frac{t}{t+1}+n)}{2^{n+1}}\,\mathrm{d}F(t)$$

(All changes of order of summation and integration are justified by the condition we put on $\omega(x)$). Now let $\omega(x) = x^L$ for some $L \in \mathbb{N}_0$ and define

$$b_s = \sum_{n=0}^\infty \frac{n^s}{2^{n+1}}.$$

Then

$$\int_0^\infty x^L\,\mathrm{d}F(x) = \int_0^\infty \sum_{i=0}^L \Big(\frac{x}{x+1}\Big)^i \binom{L}{i} b_{L-i}\,\mathrm{d}F(x),$$

whence the relation

$$M_L = \sum_{i=0}^L m_i \binom{L}{i} b_{L-i} \tag{3}$$

for $L \in \mathbb{N}_0$. The generating function of the sequence of the b_s is given by

$$b(t) = \sum_{L=0}^\infty \frac{b_L}{L!}t^L = \sum_{L=0}^\infty \sum_{n=0}^\infty \frac{n^L t^L}{2^{N+1}L!} = \sum_{n=0}^\infty \frac{e^{nt}}{2^{n+1}} = \frac{1}{2-e^t}.$$

Denote by $M(t)$ and $m(t)$ the corresponding generating functions of the coefficients M_k and m_k, respectively. Then we can rewrite (3) as

$$M(t) = \sum_{L=0}^{\infty} \frac{M_L}{L!} t^L = \frac{1}{2-e^t} \sum_{L=0}^{\infty} \frac{m_L}{L!} t^L = \frac{1}{2-e^t} m(t).$$

The function $m(t)$ is entire, and $M(t)$ has a positive radius of convergence. This already allows us to find approximate values of the moments M_L.

Theorem 6. *For $L \in \mathbb{N}_0$,*

$$\begin{aligned} M_L &= \frac{m(\log 2)}{2 \log 2} \Big(\frac{1}{\log 2} \Big)^L L! + O_\varepsilon \Big(((4\pi^2 + (\log 2)^{1/2} - \varepsilon)^{-L} \Big) L! \\ &= \Big(\frac{m(\log 2)}{2 \log 2} \Big(\frac{1}{\log 2} \Big)^L + O(6.3^{-L}) \Big) L! \end{aligned}$$

Proof. By Cauchy's formula, for any sufficiently small r,

$$M_L = \frac{L!}{2\pi i} \int_{|z|=r} \frac{M(z)}{z^{L+1}} \, \mathrm{d}z.$$

Changing the path of integration, we get by the calculus of residues

$$M_L = -\mathrm{Res}_{z=\log 2} \left(\frac{m(z)}{(2-e^z) z^{L+1}} \right) - \frac{L!}{2\pi i} \int_{|z|=R} \frac{m(z)}{2-e^z} \frac{\mathrm{d}z}{z^{L+1}},$$

where R satisfies $\log 2 < R < |\log 2 + 2\pi i|$ (which means that there is exactly one simple pole of the integrand located in the interior of the circle $|z| = R$). It is easily seen that the residue coincides with the main term in the formula of the lemma; the error term follows from estimating the integral. ■

We obtain the inverse to the linear equations (3):

$$m_L = M_L - \sum_{s=0}^{L-1} M_s \binom{L}{s}$$

for $L \in \mathbb{N}_0$. Since $b(t)(2-e^t) = 1$, the coefficients b_L can be calculated recursively

$$b_L = \sum_{s=0}^{L-1} \binom{L}{s} b_s.$$

Thus, $b_0 = 1, b_1 = 1, b_2 = 3, b_3 = 13, b_4 = 75, b_5 = 541$.

We proceed with a property of the function $m(t)$ which reflects the symmetry of the distribution function: $F(y)+F(1/y)=1$. Unfortunately, this property is still insufficient for determining the coefficients m_L. As a matter of fact,

$$m_L=\int_0^\infty\Big(\frac{x}{x+1}\Big)^L\mathrm{d}F(x) = -\int_0^\infty\Big(\frac{1/x}{1/x+1}\Big)^L\mathrm{d}F(1/x) = \int_0^\infty\Big(\frac{1}{x+1}\Big)^L\mathrm{d}F(x).$$

Since

$$\Big(\frac{x}{x+1}\Big)^L=\Big(\frac{x+1-1}{x+1}\Big)^L=\sum_{s=0}^{L}\binom{L}{s}(-1)^{L-s}\Big(\frac{1}{x+1}\Big)^{L-s},$$

this gives

$$m_L=\sum_{s=0}^{L}\binom{L}{s}(-1)^s m_s$$

for $L\geq 0$. For example, $m_1=m_0-m_1$, which gives $m_1=\frac{1}{2}$ (since $m_0=1$), and thus $M_1=\frac{3}{2}$ (see Theorem 1). For the other coefficients we only get linear relations. Thus, $2m_3=-\frac{1}{2}+3m_2$. In terms of $m(t)$ the recursion formula above yields the identity

$$m(t)=m(-t)e^t.$$

We conclude this chapter with the result, which uniquely determines the function $m(t)$ (along with the condition $m(0)=1$).

Theorem 7. *The function $m(s)$ satisfies the integral equation*

$$m(-s)=(2e^s-1)\int_0^\infty m'(-t)J_0(2\sqrt{st})\,\mathrm{d}t,\quad s\in\mathbb{R}_+,$$

where $J_0(*)$ stands for the Bessel function:

$$J_0(z)=\frac{1}{\pi}\int_0^\pi\cos(z\sin x)\,\mathrm{d}x$$

The proof of this theorem and the solution of this integral equation, and thus the explicit description of the moments will be given in a subsequent paper.

6. $p-$adic distribution

In the previous sections, we were interested in the distribution of the nth generation of the tree $\mathcal{CW}$ in the field of real numbers. Since the set of non-equivalent valuations of $\mathbb{Q}$ contains a valuation, associated with any prime number p, it is natural to consider the distribution of the set of each generation in the field of $p-$adic numbers $\mathbb{Q}_p$. In this case we have an ultrametric inequality, which implies that two circles are either co-centric, or do not intersect. We define

$$F_n(z,\nu) = 2^{-n+1}\#\{\frac{a}{b} \in \mathcal{CW}^{(n)} : \mathrm{ord}_p(\frac{a}{b} - z) \geq \nu\}, \quad z \in \mathbb{Q}_p, \quad \nu \in \mathbb{Z}.$$

(When p is fixed, the subscript p in F_n is omitted). Note that in order to calculate $F_n(z,\nu)$ we can confine to the case $\mathrm{ord}_p(z) < \nu$; otherwise $\mathrm{ord}_p(\frac{a}{b} - z) \geq \nu \Leftrightarrow \mathrm{ord}_p(\frac{a}{b}) \geq \nu$. We shall calculate the limit distribution $\mu_p(z,\nu) = \lim_{n\to\infty} F_n(z,\nu)$, and also some characteristics of it, e.g. the zeta function

$$Z_p(s) = \int\limits_{u\in\mathbb{Q}_p} |u|^s d\mu_p, \quad s \in \mathbb{C}, \quad z \in \mathbb{Q}_p,$$

where $|*|$ stands for the $p-$adic valuation.

To illustrate how the method works, we will calculate the value of F_n in two special cases. Let $p = 2$, and let $E(n)$ be the number of rational numbers in the nth generation with one of a or b being even, and let $O(n)$ be the corresponding number fractions with both a and b odd. Then $E(n)+O(n) = 2^{n-1}$. Since $\frac{a}{b}$ in the nth generation generates $\frac{a}{a+b}$ and $\frac{a+b}{b}$ in the $(n+1)$st generation, each fraction $\frac{a}{b}$ with one of the a, b even will generate one fraction with both numerator and denominator odd. If both a, b are odd, then their two offsprings will not be of this kind. Therefore, $O(n+1) = E(n)$. Similarly, $E(n+1) = E(n)+2O(n)$. This gives the recurrence $E(n+1) = E(n) + 2E(n-1)$, $n \geq 2$, and this implies

$$E(n) = \frac{2^n + 2(-1)^n}{3}, \quad O(n) = \frac{2^{n-1} + 2(-1)^{n-1}}{3}, \quad \mu_2(0,0) = \frac{2}{3}.$$

(For the last equality note that $\frac{a}{b}$ and $\frac{b}{a}$ simultaneously belong to $\mathcal{CW}^{(n)}$, and so the number of fractions with $\mathrm{ord}_2(*) > 0$ is $E(n)/2$). We will generalize this example to odd prime $p \geq 3$. Let $L_i(n)$ be the part of the fractions in the nth generations such that $ab^{-1} \equiv i \bmod p$ for $0 \leq i \leq p-1$ or $i = \infty$ (that is, $b \equiv 0 \bmod p$). Thus,

$$\sum_{i \in \mathbb{F}_p \cup \infty} L_i(n) = 1;$$

in other words, $L_i(n) = F_n(i, 1)$. For our later investigations we need a result from the theory of finite Markov chains.

Lemma 1. *Let $\boldsymbol{A}$ be a matrix of a finite Markov chain with s stages. That is, $a_{i,j} \geq 0$, and $\sum_{j=1}^{s} a_{i,j} = 1$ for all i. Suppose that $\mathbf{A}$ is irreducible (for all pairs (i,j), and some m, the entry $a_{i,j}^{(m)}$ of the matrix $\mathbf{A}^m$ is strictly positive), acyclic and recurrent (this is satisfied, if all entries of $\mathbf{A}^m$ are strictly positive for some m). Then the eigenvalue 1 is simple, if λ is another eigenvalue, then $|\lambda| < 1$, and $\mathbf{A}^m$, as $m \to \infty$, tends to the matrix $\mathbf{B}$, with entries $b_{i,j} = \pi_j$, where $(\pi_1, ..., \pi_s)$ is a unique left eigenvector with eigenvalue 1, such that $\sum_{j=1}^{s} \pi_j = 1$.*

A proof of this lemma can be found in [6], Section 3.1., Theorem 1.3.

Theorem 8. $\mu_p(z, 1) = \frac{1}{p+1}$ *for* $z \in \mathbb{Z}_p$.

Proof. Similarly as in the above example, a fraction $\frac{a}{b}$ from the nth generation generates $\frac{a}{a+b}$ and $\frac{a+b}{b}$ in the $(n+1)$st generation, and it is routine to check that

$$L_i(n+1) = \frac{1}{2} L_{\frac{i}{1-i}}(n) + \frac{1}{2} L_{i-1}(n) \quad \text{for} \quad i \in \mathbb{F}_p \cup \{\infty\}, \tag{4}$$

(Here we make a natural convention for $\frac{i}{1-i}$ and $i-1$, if $i = 1$ or ∞). In this equation, it can happen that $i - 1 \equiv \frac{i}{1-i} \bmod p$; thus, $(2i-1)^2 \equiv -3 \bmod p$. The recurrence for this particular i is to be understood in the obvious way, $L_i(n+1) = L_{i-1}(n)$. Therefore, if we denote the vector-column $(L_\infty(n), L_0(n), ..., L_{p-1}(n))^T$ by $\mathbf{v}_n$, and if $\mathcal{A}$ is a matrix of the system (4), then $\mathbf{v}_{n+1} = \mathcal{A}\mathbf{v}_n$, and hence

$$\mathbf{v}_n = \mathcal{A}^{n-1}\mathbf{v}_1,$$

where $\mathbf{v}_1 = (0, 0, 1, 0, ..., 0)^T$. In any particular case, this allows us two find the values of L_i explicitly. For example, if $p = 7$, the characteristic polynomial is

$$f(x) = \frac{1}{16}(x-1)(2x-1)(2x^2+1)(4x^4+2x^3+2x+1).$$

The list of roots is

$$\alpha_1 = 1, \quad \alpha = \frac{1}{2}, \quad \alpha_{3,4} = \pm\frac{i}{\sqrt{2}}, \quad \alpha_{5,6,7,8} = \frac{-1-\sqrt{17}}{8} \pm \frac{\sqrt{1+\sqrt{17}}}{2\sqrt{2}},$$

(with respect to the two values for the root $\sqrt{17}$), the matrix is diagonalisible, and the Jordan normal form gives the expression

$$L_i(n) = \sum_{s=1}^{8} C_{i,s}\alpha_s^n.$$

Note that the elements in each row of the $(p+1)\times(p+1)$ matrix $\mathcal{A}$ are non-negative and sum up to 1, and thus, we have a matrix of a finite Markov chain. We need to check that it is acyclic. Let $\tau(i) = i-1$, and $\sigma(i) = \frac{i}{1-i}$ for $i \in \mathbb{F}_p \cup \{\infty\}$. The entry $a_{i,j}^{(m)}$ of $\mathcal{A}^m$ is

$$a_{i,j}^{(m)} = \sum_{i_1,\dots,i_{m-1}} a_{i,i_1}\cdot a_{i_1,i_2}\cdot \ldots \cdot a_{i_{m-1},j}.$$

Therefore, we need to check that for some fixed m, the composition of m $\sigma's$ or $\tau's$ leads from any i to any j. One checks directly that for any positive k, and $i, j \in \mathbb{F}_p$,

$$\begin{aligned}
\tau^{p-1-j}\circ\sigma\circ\tau^k\circ\sigma\circ\tau^{i-1}(i) &= j,\\
\tau^{p-1-j}\circ\sigma\circ\tau^k(\infty) &= j,\\
\tau^k\circ\sigma\circ\tau^{i-1}(i) &= \infty;
\end{aligned}$$

(for $i = 0$, we write τ^{-1} for τ^{p-1}). For each pair (i, j), choose k in order the amount of compositions used to be equal (say, to m). Then obviously all entries of $\mathcal{A}^m$ are positive, ant this matrix satisfies the conditions of lemma. Since all columns also sum up to 1, $(\pi_1, ..., \pi_{p+1})$,

$\pi_j = \frac{1}{p+1}$, $1 \leq j \leq p+1$, is the needed eigenvector. This proves the theorem. ■

Theorem 9. *Let $\nu \in \mathbb{Z}$ and $z \in \mathbb{Q}_p$, and $ord_p(z) < \nu$ (or $z = 0$). Then, if z is $p-$adic integer,*

$$\mu(z,\nu) = \frac{1}{p^{\nu} + p^{\nu-1}}.$$

If z is not integer, $ord_p(z) = -\lambda < 0$,

$$\mu(z,\nu) = \frac{1}{p^{\nu+2\lambda} + p^{\nu+2\lambda-1}}.$$

For $z = 0$, $-\nu \leq 0$, we have

$$\mu(0,-\nu) = 1 - \frac{1}{p^{\nu+1} + p^{\nu}}.$$

This theorem allows the computation of the associated zeta-function:

Corollary 10. *For s in the strip $-1 < \Re s < 1$,*

$$Z_p(s) = \int_{u\in\mathbb{Q}_p} |u|^s d\mu_p = \frac{(p-1)^2}{(p-p^{-s})(p-p^s)},$$

and $Z_p(s) = Z_p(-s)$.

The proof is straightforward. It should be noted that this expression encodes all the values of $\mu(0,\nu)$ for $\nu \in \mathbb{Z}$.

Proof of Theorem 9. For shortness, when p is fixed, denote $\text{ord}_p(*)$ by $v(*)$. As before, we want a recurrence relation among the numbers $F_n(i,\kappa)$, $i \in \mathbb{Q}_+$. For each integral κ, we can confine to the case $i < p^{\kappa}$. If $i = 0$, we only consider $\kappa > 0$ and call these pairs (i,κ) "admissible". We also include $G_n(0,-\kappa)$ for $\kappa \geq 1$, where these values are defined in the same manner as F_n, only inversing the inequality, considering $\frac{a}{b} \in \mathcal{CW}^{(n)}$, such that $v(\frac{a}{b}) \leq -\kappa$; the ratio of fractions in the nth generation outside this circle. As before, a fraction $\frac{a}{b}$ in the nth generation generates the fractions $\frac{a}{a+b}$ and $\frac{a+b}{b}$ in the $(n+1)$st generation. Let $\tau(i,\kappa) = ((i-1) \bmod p^{\kappa}, \kappa)$. Then for all admissible pairs (i,κ), $i \neq 0$, the pair $\tau(i,\kappa)$ is also admissible, and

$$v(\frac{a+b}{b} - i) = \kappa \Leftrightarrow v(\frac{a}{b} - (i-1)) = \kappa.$$

Second, if $\frac{a}{a+b}=i+p^{\kappa}u$, $i\neq 1$, $u\in\mathbb{Z}_p$, and (i,κ) is admissible, then

$$\frac{a}{b}-\frac{i}{1-i}=\frac{p^{\kappa}u}{(1-i)(1-i-p^{\kappa}u)}.$$

Since $v(\frac{i}{1-i})=v(i)-v(1-i)$, this is 0 unless i is an integer, equals to $v(i)$ if the latter is >0 and equals to $-v(1-i)$ if $v(1-i)>0$. Further, this difference has valuation $\geq\kappa_0=\kappa$, if $i\in\mathbb{Z},i\not\equiv 1\bmod p$, valuation $\geq\kappa_0=\kappa-2v(1-i)$, if $i\in\mathbb{Z},i\equiv 1\bmod p$, and valuation $\geq\kappa_0=\kappa-2v(i)$ if i is not integer. In all three cases, easy to check, that, if we define $i_0=\frac{i}{1-i}\bmod p^{\kappa_0}$, the pair $\sigma(i,\kappa)=^{\text{def}}(i_0,\kappa_0)$ is admissible. For the converse, let $\frac{a}{b}=i_0+p^{\kappa_0}u$, $u\in\mathbb{Z}_p$. Then

$$\frac{a}{a+b}-\frac{i_0}{1+i_0}=\frac{p^{\kappa_0}}{(1+i_0+p^{\kappa_0}u)(1+i_0)}.$$

If $i=\frac{i_0}{1+i_0}$ is a $p-$adic integer, $i\not\equiv 1\bmod p$, this has a valuation $\geq\kappa=\kappa_0$; if i is a $p-$adic integer, $i\equiv 1(p)$, this has valuation

$$\geq\kappa=\kappa_0-2v(i_0)=\kappa_0+2v(1-i);$$

if i is not a $p-$adic integer, this has valuation

$$\geq\kappa=\kappa_0-2v(1+i_0)=\kappa_0+2v(i).$$

Thus,

$$v(\frac{a}{a+b}-i)\geq\kappa\Leftrightarrow v(\frac{a}{b}-i_0)\geq\kappa_0.$$

Let $i=1$. If $\frac{a}{a+b}=1+p^{\kappa}u$, then $\kappa>0$, $u\in\mathbb{Z}_p$, and we obtain $\frac{a}{b}=-1-\frac{1}{p^{\kappa}u}$, $v(\frac{a}{b})\leq-\kappa$. Converse is also true. Finally, for $\kappa\geq 1$,

$$v(\frac{a+b}{b})\leq-\kappa\Leftrightarrow v(\frac{a}{b})\leq-\kappa,$$

and

$$v(\frac{a}{a+b})\leq-\kappa\Leftrightarrow v(\frac{a}{b}+1)\geq\kappa.$$

Therefore, we have the recurrence relations:

$$\left\{\begin{array}{ll} F_{n+1}(i,\kappa) & =\frac{1}{2}F_n(\tau(i,\kappa))+\frac{1}{2}F_n(\sigma(i,\kappa)),\text{ if }(i,\kappa)\text{ is admissible},\\ F_{n+1}(1,\kappa) & =\frac{1}{2}F_n(0,\kappa)+\frac{1}{2}G_n(0,-\kappa),\kappa\geq 1,\\ G_{n+1}(0,-\kappa) & =\frac{1}{2}G_n(0,-\kappa)+\frac{1}{2}F_n(-1,\kappa),\kappa\geq 1.\end{array}\right. \tag{5}$$

Thus, we have an infinite matrix $\mathcal{A}$, which is a change matrix for the Markov chain. If $\mathbf{v}_n$ is an infinite vector-column of F_n's and G_n's, then $\mathbf{v}_{n+1}=\mathcal{A}\mathbf{v}_n$, and, as before, $\mathbf{v}_n=\mathcal{A}^{n-1}\mathbf{v}_1$. It is direct to check that each column also contains exactly two nonzero entries $\frac{1}{2}$, or one entry, equal to 1. In terms of Markov chains, we need to determine the classes of orbits. Then in proper rearranging, the matrix $\mathcal{A}$ looks like

$$\begin{pmatrix} \mathbf{P}_1 & 0 & \dots & 0 & \dots \\ 0 & \mathbf{P}_2 & \dots & 0 & \dots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{P}_s & 0 \\ \vdots & \vdots & \dots & 0 & \ddots \end{pmatrix},$$

where $\mathbf{P}_s$ are finite Markov matrices. Thus, we claim that the length of each orbit is finite, every orbit has a representative $G_*(0, -\kappa)$, $\kappa \geq 1$, the length of it is $p^\kappa + p^{\kappa-1}$, and the matrix is recurrent (that is, every two positions communicate). In fact, from the system above and form the expression of the maps $\tau(i, \kappa)$ and $\sigma(i, \kappa)$, the direct check shows that the complete list of the orbit of $G_*(0, -\kappa)$ consists of (and each pair of states are communicating):

$$G_*(0, -\kappa),$$
$$F_*(i, \kappa) \quad (i = 0, 1, 2, ..., p^\kappa - 1),$$
$$F_*(p^{-\lambda}u, \kappa - 2\lambda) \quad (\lambda = 1, 2, ..., \kappa - 1, u \in \mathbb{N}, u \not\equiv 0 \bmod p, u \leq p^{\kappa-\lambda}).$$

In total, we have

$$1 + p^\kappa + \sum_{\lambda=1}^{\kappa-1} (p^{\kappa-\lambda} - p^{\kappa-\lambda-1}) = p^\kappa + p^{\kappa-1}$$

members in the orbit. Thus, each $\mathbf{P}_\kappa$ in the matrix above is a finite dimensional $\ell_\kappa \times \ell_\kappa$ matrix, where $\ell_\kappa = p^\kappa + p^{\kappa-1}$. For $\kappa = 1$, the matrix $\mathbf{P}_1$ is exactly the matrix of the system (4). As noted above, the vector column $(1, 1, ..., 1)^T$ is the left eigen-vector. As in the previous theorem, it is straightforward to check that this matrix is irreducible and acyclic (that is, the entries of $\mathbf{P}_\kappa^n$ are strictly positive for sufficiently large n). In fact, since by our observation, each two members in the orbit communicate, and since we have a move $G_*(0, -\kappa) \to G_*(0, -\kappa)$, the proof of the last statement is immediate: there exists n such that any position is reachable from another in exactly n moves, and this can be achieved at the expense of the move just described. Therefore, all entries of $\mathbf{P}_\kappa^n$ are strictly positive. Thus, the claim of the theorem follows from the lemma above. ■

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