Borrower: ZGM

Lending String: *BNG,YSM,SYB,NYP,CFI

Patron:

Journal Title: Analytic number theory ; proceedings of a conference in honor of Paul T. Bateman /

Volume: **Issue:**
Month/Year: 1990 **Pages:** 451-477

Article Author:

Article Title: Bruce Reznick; Some binary partition functions

Imprint: Boston ; Birkha¨user, 1990.

ILL Number: 87315520

Location:

Borrowing Notes:Borrowing Notes; ZGM IS A FREE LENDER/SUPPLIER. WE BELONG TO IDS, METRO, SUNY LAND, AND LVIS.

Mail
Charge
Maxcost: .01IFM

Shipping Address:
LAND - CUNY Grad Library
365 5th Ave
New York, NY 10016-4309
LAND

Fax: 212-817-1604
Odyssey: 206.107.43.174
Ariel:

[3] H.Petersson, Ueber die lineare Zerlegung der den ganzen Modulformen von höherer Stufe entsprechenden Dirichletreihen in vollständige Eulersche Produkte, Acta Math. **80** (1948), 191–221.
[4] R.A.Rankin, *Modular forms and functions*, Cambridge University Press, 1977.
[5] R.A. Rankin, The adjoint Hecke operator. I, to appear in J. Madras University.
[6] R.A. Rankin, The adjoint Hecke operator. II, in: *Number theory and related topics*, Oxford University Press, 1989.
[7] B. Schoeneberg, *Elliptic modular functions*, Springer-Verlag, 1974.

Robert A. Rankin
Department of Mathematics
University of Glasgow
Glasgow G12 8QW
Scotland

Some Binary Partition Functions

BRUCE REZNICK

Dedicated to Professor Paul T. Bateman on the occasion of his retirement

1. Introduction and Overview

For $d \geq 2$, the *d-th binary partition function*, $b(d;n)$, is the number of representations

$$n = \sum_{i=0}^{\infty} \epsilon_i 2^i, \ \epsilon_i \in \{0, 1, \dots, d-1\}; \tag{1.1}$$

the usual (Euler) binary partition function is $b(\infty;n) = \lim_{d\to\infty} b(d;n)$. This paper explores various arithmetic and analytic properties of the $b(d;n)$'s. For small values of d, $b(d;n)$ is familiar:

$$b(2;n) = 1 \text{ (Euler [E1, p.333])}, \qquad (1.2)(i)$$

$$b(3;n) = s(n+1) \text{ (Thm. 5.2)}, \qquad (1.2)(ii)$$

$$b(4;n) = \lfloor n/2 \rfloor + 1 \text{ (Problem B2, 1983 Putnam [KAH])}. \qquad (1.2)(iii)$$

In (ii), $s(n)$ denotes the Stern sequence; no other $b(d;n)$'s appear in [S1].

Euler [E2, p.288] defined $b(\infty;n)$ and computed its values for $n \leq 37$. Some recurrences for $b(\infty;n)$ and, in effect, $b(2^r;n)$ were studied by Tanturri [T1,T2,T3] in the 1910s. In 1940, Mahler [M] established that $\log b(\infty;n) \sim (\log n)^2/(\log 4)$; this asymptotic estimate was refined by de Bruijn [B] in 1948. Knuth [K] also investigated the growth of $\log b(\infty;n)$ in 1966, and gave some other recurrences for $b(\infty;n)$. In 1969, Churchhouse [C4] discussed the behavior of $b(\infty;n) \pmod{2^r}$. Let $\nu_2(m)$ denote the largest power of 2 dividing m. Then 2 divides $b(\infty;n)$ for $n \geq 2$, 4 divides $b(\infty;n)$ if and only if $\nu_2(n)$ or $\nu_2(n-1)$ is positive and even, and 8 never divides $b(\infty;n)$. Churchhouse conjectured that, for all even m,

$$\nu_2(b(\infty;4m) - b(\infty;m)) = \lfloor (3\nu_2(m)+4)/2 \rfloor. \tag{1.3}$$

Author supported in part by the National Science Foundation

This conjecture was proved by Rödseth [**R5**], Gupta (thrice) [**G1,G2, G4**] and generalized by Hirschhorn and Loxton [**HL**] in 1976.

The m-ary partition function is defined by replacing 2 by m in (1.1) and eliminating the restriction on the ϵ_i's. Mahler and de Bruijn actually studied the asymptotics of the m-ary partition function. The proof of (1.3) was generalized to $m > 2$ by Rödseth, Gupta [**G3**], Andrews [**A1**] and Gupta and Pleasants [**GP**]. Restricted m-ary partition functions ($\epsilon_i = 0$ for $i \ge t$) also appeared in Gupta and Pleasants, and Dirdal [**D1,D2**]. Analysis of their generating functions shows that they are equal to the number of m-ary partitions with $\epsilon_i < m^t$ for all i (see Thm. 3.2(i) for $m = 2$.) A nice summary of this work can be found in Ch. 10.2 of [**A2**], and its exercises.

Here is the plan for the rest of the paper.

In section two, we give an infinite product representation for $F_d(x)$, the generating function of $b(d;n)$. We derive some simple relationships among the F_d's and deduce the resulting recurrences on $b(d;n)$, $b(2d;n)$ and $b(\infty;n)$, which often depend on the parity of d and n. Clearly, $b(d;n)$ is non-decreasing in d; the monotonicity in n depends on the parity of d. We show that $b(2k;2n) = b(2k;2n+1) < b(2k;2n+2)$ and that $b(2k+1;2n) \ge b(2k+1;2n+1) < b(2k+1;2n+2)$, with strict inequality in the first case if $n \ge k$. In other words, $b(2k;n)$ is an increasing staircase, and $b(2k+1;n)$ starts that way but eventually zigzags. By reducing $F_d(x)$ in $(\mathbb{Z}/2\mathbb{Z})[[x]]$, we show that $b(d;n)$ is odd if and only if n is congruent to 0 or 1 (mod d). We conclude the section with an alternate interpretation of $b(d;n)$, which was suggested to us by Richard Stanley.

In section three, we discuss the special case $d = 2^r$. We show that $F_{2^r}(x)$ is rational, and that $b(2^r;n)$ is the number of partitions of n into powers of $2 \le 2^{r-1}$. We give a closed form for $b(2^r;2^{r-1}s+t)$, $0 \le t \le 2^{r-1}-1$: it is a polynomial in s of degree $r-1$, in fact, a linear combination of $\binom{s+r-1-j}{r-1}$'s, $0 \le j \le r-1$. Each such polynomial has the same leading coefficient, so $b(2^r;n) \sim (2^{r(r-1)/2}(r-1)!)^{-1}n^{r-1}$. We conclude the section by reinterpreting some early work of Tanturri on $b(2^r;n)$.

In section four, we consider the asymptotic growth of $b(2k;n)$. We show that $b(2k;n) = \Theta(n^{\lambda(2k)})$ for $\lambda(2k) = \log_2 k$ (that is, there exist $\alpha > \beta > 0$ and n_0 so that $\alpha n^{\lambda(2k)} > b(2k;n) > \beta n^{\lambda(2k)}$ for $n \ge n_0$.) We also show that $b(2k+1;n)$ is not $\Theta(n^{\lambda(2k+1)})$ for any $\lambda(2k+1)$. (The previous result implies that $\lambda(2k+1) = \log_2(k+\frac{1}{2})$, so $b(2k+1;2^r)$ would be $\Theta((k+\frac{1}{2})^r)$. However, the recurrences imply that $b(2k+1;2^r)$ satisfies a monic linear recurrence in r with integer coefficients, and $b(2k+1;2^r) = \Theta(\tau^r)$ implies that τ is an algebraic integer–see Cor. 1.7.) We also compute $\mu_i(2k+1)$ so that, for suitable $\alpha_i > 0$, $\alpha_1 n^{\mu_1(2k+1)} > b(2k+1;n) > \alpha_2 n^{\mu_2(2k+1)}$. Since $\lambda(2k+2) > \mu_1(2k+1)$ for $k \ge 1$ and $\mu_2(2k+1) > \lambda(2k)$ for $k \ge 2$, it follows that $b(d+1;n)/b(d;n) \to \infty$ for all $d \ge 3$.

In section five, we use known properties of the Stern sequence to give more specific information about the growth of $b(3;n)$ and $b(6;n)$. We show that $\mu_1(3)$ and $\mu_2(3)$ are best possible, and that $n^{-\lambda(6)}b(6;n)$ does not converge, even though $b(6;n) = \Theta(n^{\lambda(6)})$. This is, in effect, a result of Carlitz [**C3**], which was suggested by a question of P. T. Bateman.

We conclude, in section six, with acknowledgments and some open questions, and, in an appendix, give a table of $b(d;n)$ for $0 \le n \le 32$ and $2 \le d \le 9$ and $d = \infty$.

We shall repeatedly use a familiar result on linear recurrences with constant coefficients, which goes back to Lagrange and Euler.

Linear Recurrence Theorem. *Suppose*

$$p(t) = t^r + c_1 t^{r-1} + \cdots + c_r = t^\kappa \prod_{i=1}^{s} (t-\lambda_i)^{r_i}, \tag{1.4}$$

where $c_i \in \mathbb{C}$, $0 \ne \lambda_i \in \mathbb{C}$, $r_i \ge 1$ and the λ_i's are distinct, and suppose (x_n) is a sequence satisfying the recurrence

$$x_{n+r} + c_1 x_{n+r-1} + \cdots + c_r x_n = 0, n \ge 0. \tag{1.5}$$

Then there exist polynomials h_i, of degree $r_i - 1$, so that

$$x_n = \sum_{i=1}^{s} h_i(n)\lambda_i^n \quad \text{for} \quad n \ge \kappa. \tag{1.6}$$

The simplest proof of the Linear Recurrence Theorem involves generating functions and partial fractions; one version is in [**R2**].

Corollary 1.7. *Keeping the previous notation, suppose (x_n) is a real sequence satisfying (1.5) and for some $\tau > 0$, α, $\beta > 0$ and all $n \ge n_0$,*

$$\alpha\tau_n \ge x_n \ge \beta\tau_n. \tag{1.8}$$

Then $\tau = max|\lambda_i|$ and $p(\tau) = 0$. If $p \in \mathbb{Z}[t]$, then τ is an algebraic integer.

Proof: Let $M = \max\{|\lambda_j|\}$, let $d = \max\{\deg h_j : |\lambda_j| = M\}$ and reindex so that $\lambda_j = M\epsilon_j$, $|\epsilon_j| = 1$, and $\deg h_j = d$ precisely for $1 \le j \le k$. Then, $h_j(n) = \alpha_j n^d + o(n^d)$, where $\alpha_j \ne 0$. Finally, let

$$H(n) = \sum_{j=1}^{k} \alpha_j \epsilon_n^j. \tag{1.9}$$

Then by the Linear Recurrence Theorem,

$$x_n = H(n)n^d M^n + o(n^d M^n). \tag{1.10}$$

If $|\omega| = 1$, $\omega \neq 1$, then

$$\left|\sum_{n=1}^{N} \omega^n\right| = \frac{|(\omega^{N+1} - \omega)|}{|\omega - 1|} \leq \frac{2}{|\omega - 1|} < \infty. \tag{1.11}$$

Since $|H(n)| \leq \Sigma|\alpha_j| = A$, and

$$\begin{aligned} \sum_{i=1}^{N} |H(n)|^2 &= \left(\sum_{j=1}^{k} |\alpha_j|^2\right) N + \sum_{j \neq \ell} \alpha_j \bar{\alpha}_\ell \left(\sum_{n=1}^{N} \epsilon_j^n \bar{\epsilon}_\ell^n\right) \\ &= B^2 N + \mathcal{O}(1), \end{aligned} \tag{1.12}$$

$\overline{\lim_{n\to\infty}} |H(n)| \geq B > 0$. Thus for all $\epsilon > 0$, there are infinitely many n with

$$(A + \epsilon)n^d M^n \geq x_n \geq (B - \epsilon)n^d M^n. \tag{1.13}$$

It follows from (1.8) that $d = 0$ and $M = \tau$.
Suppose $p(\tau) \neq 0$, then $\epsilon_j \neq 1$ for all j, and by (1.11),

$$\left|\sum_{n=1}^{N} H(n)\right| = \left|\sum_{j=1}^{N} \alpha_j \sum_{n=0}^{N} \epsilon_j^n\right| \leq \sum_{j=1}^{k} \frac{2|\alpha_j|}{|1 - \epsilon_j|} < \infty. \tag{1.14}$$

But by (1.8), $\overline{\lim_{n\to\infty}} H(n) \geq \beta > 0$. This is a contradiction, so $p(\tau) = 0$. ∎

2. Basic Properties of $b(d; n)$

The following infinite product formulas for the generating functions of $b(d; n)$ and $b(\infty; n)$ are immediate from (1.1):

$$\begin{aligned} F_d(x) = \sum_{n=0}^{\infty} b(d; n)x^n &= \prod_{j=0}^{\infty} (1 + x^{2^j} + \cdots + x^{(d-1)2^j}) \\ &= \prod_{j=0}^{\infty} \frac{1 - x^{d \cdot 2^j}}{1 - x^{2^j}}; \end{aligned} \tag{2.1}$$

$$F_\infty(x) = \prod_{j=0}^{\infty} \frac{1}{1 - x^{2^j}}. \tag{2.2}$$

The following theorem summarizes some elementary manipulations of the generating functions in (2.1) and (2.2).

Theorem 2.3.
(i) $F_d(x) F_\infty(x^d) = F_\infty(x)$,
(ii) $F_{2k}(x) = (1 - x)^{-1} F_k(x^2)$,
(iii) $(1 - x) F_d(x) = (1 - x^d) F_d(x^2)$,
(iv) $F_k(x) = (1 - x^k) F_{2k}(x)$.

Theorem 2.3 leads to many recurrences. For convenience, we shall construe $b(d; n)$ to be 0 when n is negative.

Theorem 2.4.
(i) $b(\infty; n) = \sum_{r=0}^{\lfloor n/d \rfloor} b(d; n - dr) b(\infty; r)$,
(ii) $b(2k; n) = \sum_{j=0}^{\lfloor n/2 \rfloor} b(k; j)$,
(iii) $b(2k; 2n) = b(2k; 2n + 1) = \sum_{j=0}^{k-1} b(2k; n - j)$,
(iv) $b(2k + 1; 2n) = \sum_{j=0}^{k} b(2k + 1; n - j)$,
(v) $b(2k + 1; 2n + 1) = \sum_{j=0}^{k-1} b(2k + 1; n - j)$,
(vi) $b(2k; n) = b(k; n) + b(2k; n - k)$,
(vii) $b(2k; n) - b(2k; n - 2) = b(k; \lfloor n/2 \rfloor)$,
(viii) $b(2k + 1; 2n) - b(2k + 1, 2n - 1) = b(2k + 1; n)$,
(ix) $b(2k + 1; 2n) - b(2k + 1, 2n + 1) = b(2k + 1; n - k)$,
(x) $b(2k; n) = \sum_{r=0}^{\lfloor n/k \rfloor} b(k; n - rk)$.

Proof: Expanding Thm. 2.3(i), we have

$$\sum_{m=0}^{\infty} b(d; m)x^m \sum_{r=0}^{\infty} b(\infty; r)x^{dr} = \sum_{n=0}^{\infty} b(\infty; n)x^n; \tag{2.5}$$

part (i) follows from comparing the coefficient of x^n on both sides of (2.5). Similarly, Thm. 2.3(ii) expands to

$$F_{2k}(x) = \sum_{n=0}^{\infty} b(2k; n)x^n = (1 + x + x^2 + \cdots) \sum_{i=0}^{\infty} b(k; i)x^{2i}, \tag{2.6}$$

which implies (ii). Thm. 2.3(iii) is equivalent to:

$$F_d(x) = \sum_{n=0}^{\infty} b(d;n)x^n = (1 + x + \cdots + x^{d-1}) \sum_{i=0}^{\infty} b(d;i)x^{2i}. \tag{2.7}$$

The term x^n occurs on the right when $n = j + 2i$, where $0 \le j \le d-1$, so $b(d;n)$ is the sum of those $b(d;n-j)$'s in which $n-j$ is an even integer. Parts (iii) through (v) arise by considering the varying parities of d and n. We obtain (vi) by writing out Thm. 2.3(iv). Finally, (vii), (viii), (ix) and (x) result from iterating (ii), (iv), (v) and (vi). ∎

Several comments about these recurrences are in order. Since $b(\infty;0) = 1$, we could use Thm. 2.4(i) to define $b(d;n)$ recursively. Also, when $r = 2$, this becomes (by (1.2)(i)),

$$b(\infty;n) = b(\infty;0) + \cdots + b(\infty;\lfloor n/2 \rfloor), \tag{2.8}$$

This equation is in Tanturri [**T2**], but also follows easily from

$$b(\infty;n) = b(\infty,n-2) + b(\infty;\lfloor n/2 \rfloor), \tag{2.9}$$

which is implicit in Euler [**E2**]. Churchhouse iterated (2.8) to express $b(\infty;2^r n)$ in terms of $\{b(\infty;j) : 0 \le j \le n\}$, and generalizations of this idea represent much of the literature on binary (and m-ary) partitions.

There are combinatorial proofs for many of these recurrences. For example, if $\epsilon_j \in \{0,\dots,2k-1\}$, then $\epsilon_j = 2\nu_j + \eta_j$, where $\nu_j \in \{0,\dots,k-1\}$ and $\eta_j \in \{0,1\}$. So, $n = \sum \epsilon_j 2^j = 2(\sum \nu_j 2^j) + (\sum \eta_j 2^j)$, and for every partition $n = 2s + t$, there are $b(k;s)\cdot 1$ ways of writing $s = \sum \nu_j 2^j$ and $t = \sum \eta_j 2^j$; this proves (ii).

We turn to the monotonicity properties of $b(d;n)$.

Theorem 2.10.
(i) $b(d;n) \le b(d+1;n)$,
(ii) $1 \le b(d;n)$,
(iii) $b(d;n) = b(\infty;n)$ *if* $d > n$,
(iv) $b(d;n) = b(\infty;n) - b(\infty;n-d)$ *if* $n \ge d > n/2$.

Proof: Any solution of (1.1) satisfies $0 \le \epsilon_i \le d-1 \le d$, whence (i); (ii) follows by induction and (1.2)(i). For (iii) and (iv), we use Thm. 2.4(i):

$$b(\infty;n) = b(d;n)b(\infty;0) + b(d;n-d)b(\infty,1) + b(d;n-2d)b(\infty;2) + \cdots. \tag{2.11}$$

Recall that $b(\infty;0) = b(\infty;1) = 1$. If $d > n$, then (2.11) implies (iii), if $n \ge d > n/2$, then $b(\infty;n) = b(d;n) + b(d;n-d)$, but $b(d;n-d) = b(\infty;n-d)$ by (iii) since $d > n-d$, thus (iv). These can also be proved combinatorially. ∎

Theorem 2.12.
(i) $b(2k;2n) = b(2k;2n+1)$,
(ii) $b(2k;2n) > b(2k;2n-1)$,
(iii) $b(2k+1;2n) > b(2k+1,2n-1)$,
(iv) $b(2k+1;2n) \ge b(2k+1,2n+1)$, *with ">" if* $n \ge k$.

Proof: Parts (i), (iii) and (iv) follow from Thm. 2.4(iii), (viii) and (ix). For (ii), Thm. 2.4(vii) implies that

$$b(2k;2n) - b(2k;2n-1) = b(2k;2n) - b(2k;2n-2) = b(k;n). \quad ∎ \tag{2.13}$$

The generating functions allow us to determine the parity of $b(d;n)$.

Theorem 2.14.

$$b(d;n) \equiv 1 \pmod 2 \iff n \equiv 0, 1 \pmod d$$

Proof: We reduce (2.1) (mod 2), viewing $F_d(x)$ as an element of $(\mathbb{Z}/2\mathbb{Z})[[x]]$:

$$F_d(x) = \prod_{j=0}^{\infty} \frac{(1 - x^{d\cdot 2^j})}{(1 - x^{2^j})} \equiv \prod_{j=0}^{\infty} \frac{(1 + x^{d\cdot 2^j})}{(1 + x^{2^j})}. \tag{2.15}$$

Since $\prod(1+x^{2^j}) = (1-x)^{-1} \equiv (1+x)^{-1}$ in $(\mathbb{Z}/2\mathbb{Z})[[x]]$, (2.15) becomes

$$\sum_{n=0}^{\infty} b(d;n)x^n = \frac{1+x}{1+x^d} \equiv (1+x)(1 + x^d + x^{2d} + \cdots). \quad ∎ \tag{2.16}$$

This is consistent with (1.2): $b(2;n) = 1$ is always odd; $b(3;n)$ is even when n is a multiple of 3 (Stern); $b(4;n)$ is odd when $\lfloor n/2 \rfloor$ is even. There is a vaguely similar formula for any prime p, based on the identity $(\varphi(x))^p = \varphi(x^p)$ for $\varphi \in (\mathbb{Z}/p\mathbb{Z})[[x]]$. Theorem 2.3(i) implies that:

$$F_p(x)(F_\infty(x))^{p-1} \equiv 1 \pmod p. \tag{2.17}$$

Let $v(m)$ denote the number of 1's in the usual (unique) binary representation of m. Richard Stanley [**S2**] has made the following interesting observation to the author.

Theorem 2.18. *Suppose ω is any primitive d-th root of unity. Then,*

$$b(d;n) = \sum (-\omega)^{v(m_1)}(-\omega^2)^{v(m_2)} \cdots (-\omega^{d-1})^{v(m_{d-1})}, \tag{2.19}$$

where the sum is taken over all (ordered) sums $n = m_1 + \cdots + m_{d-1}$.

Proof: For all z,

$$\sum_{n=0}^{\infty} z^{v(n)} x^n = \prod_{j=0}^{\infty}(1 + zx^{2^j}). \tag{2.20}$$

Since ω is a primitive d-th root of unity,

$$1 + x^{2^j} + x^{2\cdot 2^j} + \cdots + x^{(d-1)\cdot 2^j} = \prod_{\ell=1}^{d-1}(1 - \omega^\ell x^{2^j}); \tag{2.21}$$

hence by (2.1), (2.20) and (2.21),

$$\begin{aligned} F_d(x) &= \prod_{j=0}^{\infty}(1 + x^{2^j} + x^{2\cdot 2^j} + \cdots + x^{(d-1)\cdot 2^j}) \\ &= \prod_{j=0}^{\infty}\prod_{\ell=1}^{d-1}(1 - \omega^\ell x^{2^j}) = \prod_{\ell=1}^{d-1}\sum_{m=0}^{\infty}(-\omega^\ell)^{v(m)} x^m, \end{aligned} \tag{2.22}$$

from which (2.19) follows. ∎

Let $d = 3$ and $\omega = \exp(4\pi i/3)$. Then $\epsilon = -\omega = \exp(\pi i/3)$ is a primitive sixth root of 1, as is $-\omega^2 = \epsilon^{-1}$; we have $(m_1, m_2) = (j, n-j)$, and

$$b(3; n) = \sum_{j=0}^{n} \epsilon^{v(j) - v(n-j)}. \tag{2.23}$$

It follows that the positivity of $b(3;n)$ reflects upon the distribution of $\{v(j) - v(n-j)\}$ (mod 6).

3. The Case $d = 2^r$

If $d = 2^r$, then the infinite product in (2.1) telescopes:

$$F_{2^r}(x) = \sum_{n=0}^{\infty} b(2^r; n) x^n = \prod_{j=0}^{r-1} \frac{1}{1 - x^{2^j}}. \tag{3.1}$$

Theorem 3.2.

(i) $b(2^r; n)$ *is the number of partitions of* n *into* $1, 2, 2^2, \ldots, 2^{r-1}$.

(ii) $\sum_{j=0}^{2^r-1}(-1)^{v(j)} b(2^r; n-j) = 0$ *for* $n \geq 1$.

(iii) $b(2^{r+s}; n) = \sum_{j=0}^{\lfloor n/2^r \rfloor} b(2^r; n - j\cdot 2^r) b(2^s; j)$.

Proof:
(i) This is immediate from (3.1).
(ii) Note that

$$(F_{2^r}(x))^{-1} = \prod_{j=0}^{r-1}(1 - x^{2^j}) = \sum_{k=0}^{2^r-1}(-1)^{v(k)} x^k, \tag{3.3}$$

(c.f. (2.20)); (ii) follows upon multiplying both sides of (3.3) by $F_{2^r}(x)$.
(iii) Manipulation of (3.1) (or iteration of Theorem 2.3(ii) with $k = 2^s$) shows that $F_{2^{r+s}}(x) = F_{2^r}(x) F_{2^s}(x^{2^r})$, which leads directly to (iii). ∎

We remark that (i) connects $b(d;n)$ with the literature on restricted binary partitions. We may use (ii) with the Linear Recurrence Theorem to describe a closed form for $b(2^r;n)$. By (3.3), (1.6) holds with the λ_i's equal to the 2^u-th primitive roots of unity, $0 \leq u \leq r-1$; for such a λ_i, h_i has degree $r-1-u$ (see Thm. 3.6.) Finally, if r or s equals 1, then (iii) reduces to Thm. 2.4(ii) or (x). A version of (iii) seems to be in **[T2,T3]**.

We now introduce an important reparameterization for $b(2^r;n)$. For $0 \leq t \leq 2^{r-1} - 1$, let

$$f(r,t)(s) = b(2^r; 2^{r-1}s + t). \tag{3.4}$$

Using our recurrences, it is easy to compute $f(r;t)$ for small r:

$$f(1,0)(s) = b(2;s) = 1, \tag{3.5)(i}$$

$$f(2,0)(s) = b(4;2s) = f(2,1)(s) = b(4;2s+1) = s+1, \tag{3.5)(ii}$$

$$f(3,0)(s) = b(8;4s) = f(3;1)(s) = b(8;4s+1) = (s+1)^2, \tag{3.5)(iii}$$

$$f(3,2)(s) = b(8;4s+2) = f(3;3)(s) = b(8;4s+3) = (s+1)(s+2). \tag{3.5)(iv}$$

Theorem 3.6. *For all* $r \geq 1$, $0 \leq t \leq 2^{r-1} - 1$, *and* $s \geq 0$,

$$f(r,t)(s) = \sum_{j=0}^{r-2} a_j(r,t) \binom{s+r-1-j}{r-1}, \tag{3.7}$$

where the $a_j(r,t)$'s are defined recursively by

$$a_0(1,0) = a_0(2,1) = 1, \tag{3.8)(i}$$

$$a_j(r+1,2k) = a_j(r+1,2k+1) = \sum_{t=0}^{k} a_j(r,t) + \sum_{t=k+1}^{2^{r-1}-1} a_{j-1}(r,t), \tag{3.8)(ii}$$

and we take $a_{-1}(r,t) = a_{r-1}(r,t) = 0$ in (3.8)(ii) when appropriate.

Proof: Clearly, (3.7) holds for $r = 2$; assume it holds for r. Theorem 2.4(ii), rephrased and applied to $b(2^{r+1}; 2^r s + t) = f(r+1,t)(s)$ gives:

$$\begin{aligned} f(r+1,2k)(s) = f(r+1,2k+1)(s) &= \sum_{i=0}^{2^{r-1}s+k} b(2^r; i) \\ &= \sum_{t=0}^{k}\sum_{u=0}^{s} f(r,t)(u) + \sum_{t=k+1}^{2^{r-1}-1}\sum_{u=0}^{s-1} f(r,t)(u), \end{aligned} \tag{3.9}$$

By the induction hypothesis and a familiar binomial identity, we obtain:

$$\begin{aligned} &f(r+1,2k)(s) = f(r+1;2k+1)(s) \\ &= \sum_{t=0}^{k}\sum_{u=0}^{s}\sum_{j=0}^{r-2} a_j(r,t)\binom{u+r-1-j}{r-1} \\ &\quad + \sum_{t=k+1}^{2^{r-1}-1}\sum_{u=0}^{s-1}\sum_{j=0}^{r-2} a_j(r,t)\binom{u+r-1-j}{r-1} \\ &= \sum_{t=0}^{k}\sum_{j=0}^{r-2} a_j(r,t)\binom{s+r-j}{r} \\ &\quad + \sum_{t=k+1}^{2^{r-1}-1}\sum_{j=0}^{r-2} a_j(r,t)\binom{s+r-1-j}{r} \\ &\sum_{j=0}^{r-1}\left(\sum_{t=0}^{k} a_j(r,t) + \sum_{t=k+1}^{2^r-1} a_{j-1}(r,t)\right)\binom{s+r-j}{r}. \end{aligned} \tag{3.10}$$

It follows from the last expression and (3.8)(ii) that (3.7) holds for $r+1$. ∎

Since $a_0(r,t) = b(\infty;t)$, it seems unlikely that there is a closed-form expression for the $a_j(r,t)$'s. We can rephrase (3.5) in these terms:

$$f(1,0)(s) = \binom{s}{0}, \tag{3.11)(i}$$

$$f(2,0)(s) = f(2,1)(s) = \binom{s+1}{1}, \tag{3.11)(ii}$$

$$f(3,0)(s) = f(3;1)(s) = \binom{s+2}{2} + \binom{s+1}{2}, \tag{3.11)(iii}$$

$$f(3,2)(s) = f(3;3)(s) = 2\binom{s+2}{2}. \tag{3.11)(iv}$$

Corollary 3.12.
(i) For $r \ge 1$, $f(r,t)(s)$ is a polynomial in s of degree $r-1$ with leading coefficient $2^{(r-1)(r-2)/2}\{(r-1)!\}^{-1}$.

(ii)
$$\lim_{n\to\infty} \frac{b(2^r;n)}{n^{r-1}} = \frac{1}{2^{r(r-1)/2}(r-1)!}.$$

Proof:
(i) Each $\binom{s+r-1-j}{r-1}$ is a polynomial in s of degree $r-1$ with leading coefficient $\{(r-1)!\}^{-1}$. An easy induction shows that $\sum_j a_j(r,t) = 2^{(r-1)(r-2)/2}$.
(ii) Let $u = r-1$ and fix t. For $n \in \{2^{r-1}s + t : 0 \le s < \infty\}$, we have

$$n^{-u} b(2^r;n) = (2^u s + t)^{-u}\{2^{u(u-1)/2}(u!)^{-1}s^u + o(s^u)\}. \tag{3.13}$$

Since $u(u-1)/2 - u^2 = -r(r-1)/2$, (3.13) implies that (ii) holds for n in every sequence $\{2^{r-1}s + t\}$, and so for n in general. ∎

One can also prove (ii) by looking at the coefficient of $(1-z)^{-(r-1)}$ in the Laurent series for $F_{2^r}(z)$ at $z = 1$.

A. Tanturri wrote a series of papers **[T1,T2,T3]** during World War I on binary partitions. His formulas are written in the now obscure symbolic notation of Peano, and, perhaps, have not become generally known for that reason. In **[T2]**, he defines $D(2^r;n)$ to be the number of partitions of n into powers of 2 such that the largest is 2^r. There is a clear bijection between these partitions and the partitions of $n - 2r$ with parts taken from the set $\{1, 2, ..., 2^r\}$. Thus, by Thm. 3.2(i),

$$D(2^r;n) = b(2^{r+1}; n - 2^r). \tag{3.14}$$

(This also follows from $D(2^r;n) = b(2^{r+1};n) - b(2^r;n)$ and Thm. 2.4(vi).)

Proposition 3.15. *(Tanturri)*

(i) $$\sum_{r=0}^{\infty} D(2^r;n) = \sum_{r=0}^{\infty} b(2^{r+1};n-2^r) = b(\infty;n),$$

(ii) $$\sum_{r=0}^{\infty}(-1)^r D(2^r;n) = \sum_{r=0}^{\infty}(-1)^r b(2^{r+1};n-2^r) = 0 \quad \text{for} \quad n \ge 1.$$

Proof: Since every binary partition of $n \ge 1$ contains a largest power of 2, (i) is immediate. For (ii), let $E(2^r;n)$ denote the number of partitions of n into powers of 2, in which 2^r is the largest power and occurs exactly once. By replacing the unique 2^r with two 2^{r-1}'s, we obtain a partition of n in which the largest power is 2^{r-1}, which occurs more than once. Conversely, in any such partition, two 2^{r-1}'s may be coalesced into one 2^r. Thus, for $r \ge 1$, $E(2^r;n) = D(2^{r-1};n) - E(2^{r-1};n)$, and for $n \ge 2$,

$$\sum_{r=0}^{\infty}(-1)^r D(2^r;n) = \sum_{r=0}^{\infty}(-1)^r (E(2^r;n) + E(2^{r+1},n)). \tag{3.16}$$

Since $E(2^r;n) = 0$ for $r > \log_2 n$, the sum in (3.16) converges; since $E(1;n) = 0$ for $n \ge 2$, the sum is zero. ∎

4. The Growth of $b(d;n)$

In this section we discuss the growth of $b(d;n)$ as $n \to \infty$; there is a dichotomy depending on the parity of d. We have seen that $b(2^r;n) \sim c \cdot n^{r-1}$. This generalizes partially to $b(2k;n) = \Theta(n^{\log_2 k})$, but no such relation holds for $b(2k+1;n)$. These results were announced in [**R1**].

Here is a sketch of the argument. We define intervals $I_r = I_r(d)$ so that, if $2n$, $2n+1 \in I_{r+1}$, then $n - j \in I_r$, $0 \le j \le (d-1)/2$. We then use the recurrences of Thm. 2.4 (iii), (iv), (v) to estimate $b(d;n)$ on I_{r+1} in terms of $b(d;n)$ on I_r. Finally, we turn these estimates into bounds in terms of n^λ. We need two straightforward lemmas.

Lemma 4.1. *For $r \ge r_0 = \lceil \log_2 d \rceil$, let $I_r = I_r(d) = [2^r - (d-1), 2^{r+1}]$. If $2n$ or $2n+1$ belongs to I_{r+1}, then $n - j$ belongs to I_r for $0 \le j \le (d-1)/2$.*

Proof: By hypothesis, $n \le 2^{r+1}$ and

$$n - j \ge 2^r - (d-1)/2 - j \ge 2^r - (d-1). \quad ∎ \tag{4.2}$$

Lemma 4.3. *Suppose there exist γ_1, γ_2, σ, $\tau > 0$ so that for $n \in I_r$, $r \ge r_0$,*

$$\gamma_1 \sigma^r \ge b(d;n) \ge \gamma_2 \tau^r. \tag{4.4}$$

Then there exist new constants $\delta_i > 0$ so that for all sufficiently large n,

$$\delta_1 n^{\log_2 \sigma} \ge b(d;n) \ge \delta_2 n^{\log_2 \tau}. \tag{4.5}$$

Proof: If $r \ge r_0 + 1$, then $2^r - (d-1) \ge 2^{r-1}$, so if $n \in I_r$, then $2^{r+1} \ge n \ge 2^{r-1}$. Since $\rho = (2^r)^{\log_2 \rho}$ for any $\rho > 0$, it follows that

$$\rho n^{\log_2 \rho} = (2n)^{\log_2 \rho} \ge (2^r)^{\log_2 \rho} \ge (n/2)^{\log_2 \rho} = \rho^{-1} n^{\log_2 \rho}. \tag{4.6}$$

Thus, (4.6) with $\rho = \sigma$ and τ, and (4.4) combine to give (4.5). ∎

Theorem 4.7. *For all $k \ge 1$, $b(2k;n) = \Theta(n^{\lambda(2k)})$, where*

$$\lambda(2k) = \log_2 k. \tag{4.8}$$

Proof: We must find α and $\beta > 0$ and n_0 so that

$$\alpha n^{\lambda(2k)} \ge b(2k;n) \ge \beta n^{\lambda(2k)}, n \ge n_0. \tag{4.9}$$

By Lemma 4.3, it suffices to show that for $r \ge r_0$ there exist $\gamma_i > 0$ so that:

$$\gamma_1 k^r \ge b(2k;n) \ge \gamma_2 k^r \quad \text{for} \quad n \in I_r. \tag{4.10}$$

Let

$$M(2k;r) = \max\{b(2k;n) : n \in I_r\}, \tag{4.11)(i}$$
$$m(2k;r) = \min\{b(2k;n) : n \in I_r\}. \tag{4.11)(ii}$$

In Thm. 2.4(iii), if the argument on the left hand side comes from I_{r+1}, then the arguments on the right hand side come from I_r by Lemma 4.1. Thus, for $n \in I_{r+1}$:

$$kM(2k;r) \ge b(2k;n) \ge km(2k;r). \tag{4.12}$$

Taking the maximum and minimum in (4.12) for $n \in I_{r+1}$, we have

$$kM(2k;r) \ge M(2k;r+1), \tag{4.13)(i}$$
$$m(2k;r+1) \ge km(2k;r). \tag{4.13)(ii}$$

It follows from (4.13) that, for $n \in I_r$,

$$\begin{aligned} M(2k;r_0)k^{r-r_0} \ge M(2k;r) \ge b(2k;n) \ge m(2k;r) \\ \ge m(2k;r_0)k^{r-r_0}. \end{aligned} \tag{4.14}$$

This is an inequality of shape (4.10), which completes the proof. ∎

One fundamental difference between the even and the odd case is that, in Thm. 2.4(iv) and (v), the number of terms in the recurrences for $b(2k+1;2n)$ and $b(2k+1;2n+1)$ depend on the parity of n and k. The proof of the following theorem is quite oblique, and a more direct proof would be desirable.

Theorem 4.15. *There do not exist ν, α and $\beta > 0$ so that for $n \ge N$,*

$$\alpha n^{\nu} \ge b(2k+1;n) \ge \beta n^{\nu}. \tag{4.16}$$

Proof: Suppose to the contrary, that (4.16) holds, and let

$$R = \sum_{j=0}^{N-1} b(2k+1;j). \tag{4.17}$$

Then by Thm. 2.4(ii), for $n \ge N$, we would have

$$R + \alpha \sum_{j=N}^{n} j^{\nu} \ge \sum_{j=0}^{n} b(2k+1;j) = b(4k+2;2n) \ge R + \beta \sum_{j=N}^{n} j^{\nu}. \tag{4.18}$$

By the usual estimates for $\sum j^{\nu}$, (4.18) implies that, for suitable constants c_i and sufficiently large n,

$$c_1 + c_2 n^{\nu+1} \ge b(4k+2;2n) \ge c_3 + c_4 n^{\nu+1}. \tag{4.19}$$

In view of Thm. 4.7, it follows that

$$\nu + 1 = \lambda(4k+2) = \log_2(2k+1). \tag{4.20}$$

Thus, for t sufficiently large, (4.16) and (4.20) imply that

$$\alpha(k+\frac{1}{2})^t \ge b(2k+1;2^t) \ge \beta(k+\frac{1}{2})^t. \tag{4.21}$$

Let $M = M_{2k+1} = [m_{ij}]$, $0 \le i, j \le 2k$ denote the matrix in which

$$m_{ij} = 1 \quad \text{if} \quad \lceil i/2 \rceil \le j \le \lfloor i/2 \rfloor + k, \; m_{ij} = 0 \quad \text{otherwise}. \tag{4.22}$$

Thus, the even rows of M contain a block of $(k+1)$ 1's and the odd rows contain a block of k 1's, and these blocks sidestep their way from northwest to southeast. Define the $(2k+1)$-column vector V_t by:

$$V_t = (b(2k+1;2^t), b(2k+1;2^t-1), ..., b(2k+1;2^t-2k))^T. \tag{4.23}$$

Then by construction, and Thm. 2.4 (iv), (v),

$$V_{t+1} = MV_t, \quad t \ge 0. \tag{4.24}$$

Hence for $t \ge 0$,

$$V_t = M^t V_0. \tag{4.25}$$

Let

$$p(t) = det[xI - M] = x^{2k+1} + c_1 x^{2k} + \cdots + c_{2k+1} \tag{4.26}$$

denote the characteristic polynomial of M. Then $p \in \mathbb{Z}[x]$ and, by the Cayley-Hamilton theorem, $p(M) = 0$. Thus, for $t \ge 0$,

$$M^{t+2k+1} + c_1 M^{t+2k} + \cdots + c_{2k+1} M^t = 0. \tag{4.27}$$

It follows from (4.25) that for $t \ge 0$,

$$V_{t+2k+1} + c_1 V_{t+2k} + \cdots + c_{2k+1} V_t = (0, 0, \ldots, 0)^T. \tag{4.28}$$

Let $x_t = b(2k+1;2^t)$; taking the first component of (4.28), we obtain

$$x_{t+2k+1} + c_1 x_{t+2k} + \cdots + c_{2k+1} x_t = 0 \quad \text{for} \quad t \ge 0. \tag{4.29}$$

That is, (1.5) holds for (x_t). But by Cor. 1.7, (4.21) and (4.29) imply that $k + \frac{1}{2}$ is an algebraic integer. This contradiction completes the proof. ∎

In any event, the monotonicity of $b(d;n)$ in d implies that, for suitable constants and sufficiently large n,

$$\alpha n^{\log_2(k+1)} \ge b(2k+2;n) \ge b(2k+1;n) \ge b(2k;n) \ge \beta n^{\log_2 k}. \tag{4.30}$$

We can improve on (4.30) by using a lemma, which we do not give in its greatest generality.

Lemma 4.31. *Suppose $M = [a_{ij}]$ is a real 2×2 matrix, $a_{ij} > 0$, with eigenvalue $\lambda > 0$, and associated eigenvector (v_1, v_2), $v_i > 0$. Suppose for all $r \ge 0$, the sequences $(f_i(r))$ and $(h_i(r))$ satisfy the inequalities:*

$$f_1(r) \ge f_2(r) > 0, \tag{4.32)(i}$$
$$f_1(r+1) \le a_{11} f_1(r) + a_{21} f_2(r), \tag{4.32)(ii}$$
$$f_2(r+1) \le a_{12} f_1(r) + a_{22} f_2(r); \tag{4.32)(iii}$$

$$h_1(r) \ge h_2(r) > 0, \tag{4.33)(i}$$
$$h_1(r+1) \ge a_{11} h_1(r) + a_{21} h_2(r), \tag{4.33)(ii}$$
$$h_2(r+1) \ge a_{12} h_1(r) + a_{22} h_2(r). \tag{4.33)(iii}$$

Then there exist $c > 0$ and $c' > 0$ so that for all $r \geq 0$,

$$c\lambda^r \geq f_1(r) \geq f_2(r), \tag{4.34}$$
$$h_1(r) \geq h_2(r) \geq c'\lambda^r. \tag{4.35}$$

Proof: First, we take the (v_1, v_2) linear combination of (4.32)(ii) and (iii), which preserves the inequality. Since $(v_1, v_2)^T$ is an eigenvector,

$$\begin{aligned} &v_1 f_1(r+1) + v_2 f_2(r+1) \\ &\leq (v_1 a_{11} + v_2 a_{12}) f_1(r) + (v_1 a_{21} + v_2 a_{22}) f_2(r) \\ &= \lambda(v_1 f_1(r) + v_2 f_2(r)) \end{aligned} \tag{4.36}$$

Since $f_2(r) > 0$, (4.36) iterated r times implies that

$$v_1 f_1(r) \leq v_1 f_1(r) + v_2 f_2(r) \leq (v_1 f_1(0) + v_2 f_2(0))\lambda^r. \tag{4.37}$$

Thus, (4.34) holds with $c = f_1(0) + v_1^{-1} v_2 f_2(0) > 0$. Similar reasoning applied to (4.33) leads to

$$v_1 h_1(r) + v_2 h_2(r) \geq (v_1 h_1(0) + v_2 h_2(0))\lambda^r, \tag{4.38}$$

and, since $h_1(r) \geq h_2(r)$, (4.38) implies that

$$(v_1 + v_2) h_1(r) \geq (v_1 + v_2) h_2(0)\lambda^r. \tag{4.39}$$

By (4.33)(iii) and (4.39),

$$h_2(r+1) \geq a_{12} h_1(r) + a_{22} h_2(r) \geq a_{12} h_1(r) \geq a_{12} h_2(0)\lambda^r. \tag{4.40}$$

Thus, (4.35) holds for $c' = \min\{h_2(0), a_{12} h_2(0)\lambda^{-1}\} > 0$. ∎

Theorem 4.41. *There exist $\mu_i(2k+1)$ and α and $\beta > 0$ so that for $n \geq n_0$,*

$$\alpha n^{\mu_1(2k+1)} \geq b(2k+1; n) \geq \beta n^{\mu_2(2k+1)}. \tag{4.42}$$

Moreover,

$$\lambda(2k+2) > \mu_1(2k+1),\ k \geq 1 \qquad (4.43)(\text{i})$$
$$\mu_2(2k+1) > \lambda(2k),\ k \geq 2. \qquad (4.43)(\text{ii})$$

Proof: We mimic the proof of Thm. 4.7. Let

$$M^e(2k+1; r) = \max\{b(2k+1; n) : n \in I_r,\ n \text{ even}\}, \qquad (4.44)(\text{i})$$
$$M^o(2k+1; r) = \max\{b(2k+1; n) : n \in I_r,\ n \text{ odd}\}, \qquad (4.44)(\text{ii})$$
$$m^e(2k+1; r) = \min\{b(2k+1; n) : n \in I_r,\ n \text{ even}\}, \qquad (4.44)(\text{iii})$$
$$m^o(2k+1; r) = \min\{b(2k+1; n) : n \in I_r,\ n \text{ odd}\}. \qquad (4.44)(\text{iv})$$

By Thm. 2.12(iii) and (iv), $b(2k+1; 2m) \geq b($

$$M^e(2k+1; r) \geq M^o(\quad\ldots\quad ; r),$$
$$m^e(2k+1; r) \geq m(\quad\ldots \qquad 4.59)(\text{i})$$

468 Observe th

As before, (4.42) follows if we can find c_i … hav …)(ii)

$$c_1 \tau^r \geq M^e(2k+1; r),$$
$$m^o(2k+1; r) \geq c_2 \sigma r.$$

We divide into two cases, depending on $k \pmod 2$.

First suppose $k = 2s$, $s \geq 1$. Then, Thm. 2.4 (iv), (v) becomes

$$b(4s+1; 2n) = b(4s+1; n) + \cdots + b(4s+1; n-2s), \qquad (4.47)(\text{i})$$
$$b(4s+1; 2n+1) = b(4s+1; n) + \cdots + b(4s+1; n-(2s-1)). \qquad (4.47)(\text{ii})$$

There are $2s+1$ $(n-j)$'s on the right hand side of (4.47)(i), either $s+1$ or s of them are even, and the rest odd. Taking the most extreme cases, we obtain the following estimates for $2n \in I_{r+1}$:

$$b(4s+1; 2n) \leq (s+1)M^e(4s+1; r) + sM^o(4s+1; r), \qquad (4.48)(\text{i})$$
$$b(4s+1; 2n) \geq sm^e(4s+1; r) + (s+1)m^o(4s+1; r). \qquad (4.48)(\text{ii})$$

Similarly, there are $2s$ $(n-j)$'s on the right-hand side of (4.47)(ii), so s of them are even and s are odd, and

$$b(4s+1; 2n+1) \leq sM^e(4s+1; r) + sM^o(4s+1; r), \qquad (4.49)(\text{i})$$
$$b(4s+1; 2n+1) \geq sm^e(4s+1; r) + sm^o(4s+1; r). \qquad (4.49)(\text{ii})$$

In (4.48) and (4.49), we take the maximum over $2n$, $2n+1 \in I_{r+1}$ in (i) and the minimum in (ii) and, in view of (4.45), obtain two systems like (4.32) and (4.33), (with f_1, $f_2 = M^e$, M^o and h_1, $h_2 = m^e, m^o$):

$$M^e(4s+1; r+1) \leq (s+1)M^e(4s+1; r) + sM^o(4s+1; r), \qquad (4.50)(\text{i})$$
$$M^o(4s+1; r+1) \leq sM^e(4s+1; r) + sM^o(4s+1; r); \qquad (4.50)(\text{ii})$$

$$m^e(4s+1; r+1) \geq sm^e(4s+1; r) + (s+1)m^o(4s+1; r), \qquad (4.51)(\text{i})$$
$$m^o(4s+1; r+1) \geq sm^e(4s+1; r) + sm^o(4s+1; r). \qquad (4.51)(\text{ii})$$

Then there exist $c > 0$ and $c' > 0$ so that for all $r \geq 0$,

$$c\lambda^r \geq f_1(r) \geq f_2(r), \tag{4.34}$$
$$h_1(r) \geq h_2(r) \geq c'\lambda^r. \tag{4.35}$$

Proof: First, we take the (v_1, v_2) linear combination of (4.32)(ii) and (iii), which preserves the inequality. Since $(v_1, v_2)^T$ is an eigenvector,

$$\begin{aligned} &v_1 f_1(r+1) + v_2 f_2(r+1) \\ &\leq (v_1 a_{11} + v_2 a_{12}) f_1(r) + (v_1 a_{21} + v_2 a_{22}) f_2(r) \\ &= \lambda(v_1 f_1(r) + v_2 f_2(r)) \end{aligned} \tag{4.36}$$

Since $f_2(r) > 0$, (4.36) iterated r times implies that

$$v_1 f_1(r) \leq v_1 f_1(r) + v_2 f_2(r) \leq (v_1 f_1(0) + v_2 f_2(0))\lambda^r. \tag{4.37}$$

Thus, (4.34) holds with $c = f_1(0) + v_1^{-1} v_2 f_2(0) > 0$. Similar reasoning applied to (4.33) leads to

$$v_1 h_1(r) + v_2 h_2(r) \geq (v_1 h_1(0) + v_2 h_2(0))\lambda^r, \tag{4.38}$$

and, since $h_1(r) \geq h_2(r)$, (4.38) implies that

$$(v_1 + v_2) h_1(r) \geq (v_1 + v_2) h_2(0) \lambda^r. \tag{4.39}$$

By (4.33)(iii) and (4.39),

$$h_2(r+1) \geq a_{12} h_1(r) + a_{22} h_2(r) \geq a_{12} h_1(r) \geq a_{12} h_2(0) \lambda^r. \tag{4.40}$$

Thus, (4.35) holds for $c' = \min\{h_2(0), a_{12} h_2(0) \lambda^{-1}\} > 0$. ∎

Theorem 4.41. *There exist $\mu_i(2k+1)$ and α and $\beta > 0$ so that for $n \geq n_0$,*

$$\alpha n^{\mu_1(2k+1)} \geq b(2k+1; n) \geq \beta n^{\mu_2(2k+1)}. \tag{4.42}$$

Moreover,

$$\lambda(2k+2) > \mu_1(2k+1), k \geq 1 \qquad (4.43)(i)$$
$$\mu_2(2k+1) > \lambda(2k), k \geq 2. \qquad (4.43)(ii)$$

Proof: We mimic the proof of Thm. 4.7. Let

$$M^e(2k+1; r) = \max\{b(2k+1; n) : n \in I_r,\ n \text{ even}\}, \qquad (4.44)(i)$$
$$M^o(2k+1; r) = \max\{b(2k+1; n) : n \in I_r,\ n \text{ odd}\}, \qquad (4.44)(ii)$$
$$m^e(2k+1; r) = \min\{b(2k+1; n) : n \in I_r,\ n \text{ even}\}, \qquad (4.44)(iii)$$
$$m^o(2k+1; r) = \min\{b(2k+1; n) : n \in I_r,\ n \text{ odd}\}. \qquad (4.44)(iv)$$

By Thm. 2.12(iii) and (iv), $b(2k+1; 2m) \geq b(2k+1; 2m \pm 1)$, hence

$$M^e(2k+1; r) \geq M^o(2k+1; r), \qquad (4.45)(i)$$
$$m^e(2k+1; r) \geq m^o(2k+1; r). \qquad (4.45)(ii)$$

As before, (4.42) follows if we can find $c_i > 0$ and σ, τ so that for $r \geq r_0$,

$$c_1 \tau^r \geq M^e(2k+1; r), \qquad (4.46)(i)$$
$$m^o(2k+1; r) \geq c_2 \sigma r. \qquad (4.46)(ii)$$

We divide into two cases, depending on k (mod 2).

First suppose $k = 2s$, $s \geq 1$. Then, Thm. 2.4 (iv), (v) becomes

$$b(4s+1; 2n) = b(4s+1; n) + \cdots + b(4s+1; n-2s), \qquad (4.47)(i)$$
$$b(4s+1; 2n+1) = b(4s+1; n) + \cdots + b(4s+1; n-(2s-1)). \qquad (4.47)(ii)$$

There are $2s+1$ $(n-j)$'s on the right hand side of (4.47)(i), either $s+1$ or s of them are even, and the rest odd. Taking the most extreme cases, we obtain the following estimates for $2n \in I_{r+1}$:

$$b(4s+1; 2n) \leq (s+1) M^e(4s+1; r) + s M^o(4s+1; r), \qquad (4.48)(i)$$
$$b(4s+1; 2n) \geq s m^e(4s+1; r) + (s+1) m^o(4s+1; r). \qquad (4.48)(ii)$$

Similarly, there are $2s$ $(n-j)$'s on the right-hand side of (4.47)(ii), so s of them are even and s are odd, and

$$b(4s+1; 2n+1) \leq s M^e(4s+1; r) + s M^o(4s+1; r), \qquad (4.49)(i)$$
$$b(4s+1; 2n+1) \geq s m^e(4s+1; r) + s m^o(4s+1; r). \qquad (4.49)(ii)$$

In (4.48) and (4.49), we take the maximum over $2n$, $2n+1 \in I_{r+1}$ in (i) and the minimum in (ii) and, in view of (4.45), obtain two systems like (4.32) and (4.33), (with f_1, $f_2 = M^e$, M^o and h_1, $h_2 = m^e, m^o$):

$$M^e(4s+1; r+1) \leq (s+1) M^e(4s+1; r) + s M^o(4s+1; r), \qquad (4.50)(i)$$
$$M^o(4s+1; r+1) \leq s M^e(4s+1; r) + s M^o(4s+1; r); \qquad (4.50)(ii)$$

$$m^e(4s+1; r+1) \geq s m^e(4s+1; r) + (s+1) m^o(4s+1; r), \qquad (4.51)(i)$$
$$m^o(4s+1; r+1) \geq s m^e(4s+1; r) + s m^o(4s+1; r). \qquad (4.51)(ii)$$

Observe that the matrices

$$M_1 = \begin{bmatrix} s+1 & s \\ s & s \end{bmatrix}, \quad M_2 = \begin{bmatrix} s & s \\ s+1 & s \end{bmatrix}, \tag{4.52}$$

have characteristic equations

$$p_1(x) = x^2 - (2s+1)x + s,\ p_2(x) = x^2 - 2sx - s, \tag{4.53}$$

respectively. We choose their larger eigenvalues:

$$\lambda_1 = \lambda_1(s) = \frac{1}{2}((2s+1) + (4s^2+1)^{1/2}), \tag{4.54)(i}$$
$$\lambda_2 = \lambda_2(s) = s + (s^2+s)^{1/2}. \tag{4.54)(ii}$$

For $s \geq 1$, each row of each $M_i - \lambda_i I$ has one positive and one negative entry, so the associated eigenvector has positive components. Thus the hypotheses of Lemma 4.31 are satisfied, upon identifying (4.50) and (4.51) with (4.32) and (4.33). We conclude from (4.34) and (4.35) that:

$$M^o(4s+1;r) \leq M^e(4s+1;r) \leq c\lambda_1^r, \tag{4.55)(i}$$
$$m^e(4s+1;r) \geq m^o(4s+1;r) \geq c'\lambda_2^r. \tag{4.55)(ii}$$

By the previous argument, it follows that (4.42) holds with

$$\mu_1(4s+1) = \log_2 \lambda_1(s),\ \mu_2(4s+1) = \log_2 \lambda_2(s). \tag{4.56}$$

We wish to establish (4.43) in this case. Considering (4.8), (4.54) and (4.56), we exponentiate both sides of (4.43) to base 2, obtaining

$$2s+1 > \frac{1}{2}((2s+1) + (4s^2+1)^{1/2}),\ s \geq 1, \tag{4.57)(i}$$
$$s + (s^2+s)^{1/2} > 2s,\ s \geq 1. \tag{4.57)(ii}$$

These inequalities may be routinely verified, completing the proof.

The identical reasoning holds when $k = 2s+1$, $s \geq 0$, with slight numerical changes. We have

$$b(4s+3;2n) = b(4s+1;n) + \cdots + b(4s+1;n-2s-1), \tag{4.58)(i}$$
$$b(4s+3;2n+1) = b(4s+1;n) + \cdots + b(4s+1;n-2s). \tag{4.58)(ii}$$

Arguing as before, but without the details, we find that

$$M^e(4s+3;r+1) \leq (s+1)M^e(4s+3;r) + (s+1)M^o(4s+3;r), \tag{4.59)(i}$$
$$M^o(4s+3;r+1) \leq (s+1)M^e(4s+3;r) + sM^o(4s+3;r); \tag{4.59)(ii}$$

$$m^e(4s+3;r+1) \geq (s+1)m^e(4s+3;r) + (s+1)m^o(4s+3;r), \tag{4.60)(i}$$
$$m^o(4s+3;r+1) \geq sm^e(4s+3;r) + (s+1)m^o(4s+3;r). \tag{4.60)(ii}$$

The matrices

$$M_3 = \begin{bmatrix} s+1 & s+1 \\ s+1 & s \end{bmatrix}, \quad M_4 = \begin{bmatrix} s+1 & s \\ s+1 & s+1 \end{bmatrix}, \tag{4.61}$$

again satisfy the hypotheses of Lemma 4.31 with eigenvalues

$$\lambda_3 = \lambda_3(s) = \frac{1}{2}((2s+1) + (4s^2+8s+5)^{1/2}), \tag{4.62)(i}$$
$$\lambda_4 = \lambda_4(s) = s+1+(s^2+s)^{1/2}, \tag{4.62)(ii}$$

and we conclude that (4.42) holds, where

$$\mu_1(4s+3) = \log_2 \lambda_3(s),\ \mu_2(4s+3) = \log_2 \lambda_4(s). \tag{4.63}$$

Again, verification of (4.43) reduces to two more easy inequalities:

$$2s+2 > \frac{1}{2}((2s+1) + (4s^2+8s+5)^{1/2}),\ s \geq 0, \tag{4.64)(i}$$
$$s+1+(s^2+s)^{1/2} > 2s+1,\ s \geq 1. \ \blacksquare \tag{4.64)(ii}$$

No claim is made that $\mu_1(2k+1)$ and $\mu_2(2k+1)$ are best possible for all k, although we show that this is true for $k = 1$ (see Thm. 5.13.)

Corollary 4.65. *If $d \geq 3$, then*

$$\lim_{n\to\infty} \frac{b(d+1;n)}{b(d;n)} = \infty.$$

The omission of $d = 2$ is intentional. It is easy to check that $b(3;2^r-1) = 1$ for all r, hence $b(3;n)/b(2;n) = 1$ infinitely often.

5. Two Special Cases

In this section we discuss in greater detail the growth of $b(3;n)$ and $b(6;n)$. We have seen that $b(2r;n)$ is very well-behaved. On the other hand, $b(3;n)$ is quite irregular, and its growth is described most easily in terms of a closely related sequence.

The Stern sequence was first studied [**S3**] in the 1850s by M. Stern, a student of Eisenstein, and has reappeared sporadically in the literature. (See [**R2**] for a more extensive bibliography.) It is defined recursively:

$$s(0) = 0, s(1) = 1, s(2n) = s(n), s(2n+1) = s(n) + s(n+1), n \geq 1. \quad (5.1)$$

The Stern sequence, *per se.* was apparently first defined in de Rham [**R4**]. The block of terms $\{s(2^r), s(2^r+1), \ldots, s(2^{r+1})\}$ formed the r-th row in the Stern-Brocot array, which was studied by Stern, Lucas [**L3**], Lehmer [**L1**], et al. We construe results about these terms as results about the Stern sequence.

Theorem 5.2.

$$b(3;n) = s(n+1) \quad \text{for} \quad n \geq 0. \quad (5.3)$$

Proof: By Thm. 2.4(iv), (v), we have the recurrences:

$$b(3;2n) = b(3;n) + b(3;n-1); b(3;2n+1) = b(3;n) \quad \text{for} \quad n \geq 1. \quad (5.4)$$

Together with the initial condition $b(3;0) = 1$, (5.4) determines $b(3;n)$ for all n. A comparison with (5.1) shows that (5.3) holds for $n \leq 1$. An easy induction now shows that it holds for all n:

$$b(3;2n) = b(3;n) + b(3;n-1) = s(n+1) + s(n) = s(2n+1), \quad (5.5)(i)$$

$$b(3;2n+1) = b(3;n) = s(n+1) = s(2n+2). \blacksquare \quad (5.5)(ii)$$

An incorrect version of Thm. 5.2 appeared in [**C1,C2,C3,L2**]. Carlitz defined $\theta_0(n)$ to be the number of odd Stirling numbers $S(n,2r)$ of the second kind, and proved it was also the number of odd binomial coefficients $\binom{t}{u}$ so that $t+u=n$. He showed that $\theta_0(n)$ satisfied the same recurrences as $b(3;n)$ and gave its generating function:

$$G(x) = \sum_{n=0}^{\infty} \theta_0(n) x^n = \prod_{j=0}^{\infty} (1 + x^{2^j} + x^{2^{j+1}}). \quad (5.6)$$

Thus, $G(x) = F_3(x)$ and $\theta_0(n) = b(3;n)$. From this, "it is clear that $\theta_0(n)$ is the number of partitions

$$n = n_0 + n_1 \cdot 2 + n_2 \cdot 2^2 + \cdots \quad (0 \leq n_j \leq 2) \quad (5.7)$$

subject to the following conditions: if $n_0 = 1$, then $n_1 \leq 1$, if $n_1 = 2$, then $n_2 \leq 1$, if $n_2 = 2$, then $n_3 \leq 1$, and so on." [**C1**, p.62]. Since $\theta_0(n)$ is, in fact, the number of such partitions without the given conditions, there is an error. Apparently, (5.6) was viewed as counting partitions of n in which the j-th part was chosen from $\{0, 2^j, 2^{j+1}\}$. If $n_0 = 1$, then 2^0 was chosen in the 0-th part, and there is only one part left in which to select 2^1, etc. The error first presents itself at $n = 5$, where $(n_0, n_1, n_2) = (1,2,0)$ or $(1,0,1)$. Note that the first violates the "conditions", but the second occurs twice in this alternate interpretation: $5 = 2^0 + 2^2$ and 2^0 is taken in the 0-th part, but 2^2 may be taken either in the first or second part.

We need the following classical facts about the Stern sequence.

Proposition 5.8. *(Lucas, Lehmer) For $r \geq 0$, let $I_r = [2^r, 2^{r+1}]$, and define $M_r = \max\{s(n) : n \in I_r\}$ and $m_r = \min\{s(n) : n \in I_r\}$ Then*

$$M_r = F_{r+2}, m_r = 1, \quad (5.9)$$

where F_n is the n-th Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$.)

Proof: Since $s(2^r) = 1$ for all r and $s(n) = b(3;n-1) \geq 1$, $m_r = 1$. By (5.1), $s(2n \pm 1) - s(2n) = s(n \pm 1)$, so $M_r = s(m)$, where m is odd. As $s(4n+1) = s(2n+1) + s(n)$ and $s(4n+3) = s(2n+1) + s(n+1)$,

$$M_r \leq M_{r-1} + M_{r-2}. \quad (5.10)$$

We see from Table 7.1 that $M_0 = 1$, $M_1 = 2$, $M_2 = 3$, and $M_3 = 5$. Define the sequence a^r by $a_{2r} = (2^{2r+2} - 1)/3$ and $a_{2r+1} = (2^{2r+3} + 1)/3$. (That is, a_t is the integer closest to $\frac{4}{3}2^t$.) Then $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, $a_3 = 11$, etc., and $a_t \in I_t$. We wish to show that $M_t = s(a_t)$ for all t. From their definitions, $a_{2r} = 2a_{2r-1} - 1$ and $a_{2r+1} = 2a_{2r} + 1$, hence

$$s(a_{2r}) = s(a_{2r-1}) + s(a_{2r-1} - 1) = s(a_{2r-1}) + s(a_{2r-2}), \quad (5.11)(i)$$

$$s(a_{2r+1}) = s(a_{2r}) + s(a_{2r} + 1) = s(a_{2r}) + s(a_{2r-1}). \quad (5.11)(ii)$$

Since $M_r = s(a_r)$ for $r \leq 3$, by (5.10) and (5.11),

$$M_{r-1} + M_{r-2} \geq M_r \geq s(a_r) = s(a_{r-1}) + s(a_{r-2}) = M_{r-1} + M_{r-2}. \quad (5.12)$$

Since $M_0 = 1 = F_2$ and $M_1 = 2 = F_3$, it follows that $M_t = F_{t+2}$. $\blacksquare$

It can be shown (see [**R2**]) that, for $0 \leq k \leq 2^r$, $s(2^r + k) = s(2^{r+1} - k)$. Thus, $M_r = F_{r+2}$ also equals $s(b_r)$, where b_r is the integer closest to $\frac{5}{3}2^r$.

Theorem 5.13. *Let $\varphi = \frac{1}{2}(1+\sqrt{5})$ and $\mu = \log_2 \varphi$. Then for $n \ge 1$, and some $c > 0$,*

$$cn^\mu \ge b(3;n) \ge 1, \tag{5.14}$$

and $\mu = \mu_1(3)$ is best possible.

Proof: The lower bound is clear since $b(3;2^r-1) = s(2^r) = 1$. By Thm. 5.2 and Prop. 5.8, $b(3;n) \le F_{r+1}$ for $n \le 2^r - 1$. The Binet formula for the Fibonacci numbers implies that $F_{r+1} = (\varphi/\sqrt{5})\varphi^r + o(1)$, hence

$$b(3;n) \le (\varphi/\sqrt{5})\varphi^r + o(1) \le (\varphi/\sqrt{5})n^\mu + o(1). \tag{5.15}$$

On the other hand, in the notation of the last theorem,

$$b(3;a_r-1) = F_{r+2} = (\varphi^2/\sqrt{5})\varphi^r + o(1). \tag{5.16}$$

Since $a_r - 1 \approx \frac{4}{3}2^r$, the constant μ cannot be reduced. ∎

Theorem 5.17. *Suppose $u = 2t$ is even. Then for $r \ge 0$,*

$$b(6;u\cdot 2^r-1) = (b(6;u-1) - \frac{1}{2}b(3;u-1))3^r + \frac{1}{2}b(3;u-1). \tag{5.18}$$

Proof: Let $A_r = b(6;u\cdot 2^r - 1)$. Then by Thm. 2.4(ii), (iv) and (v),

$$\begin{aligned} A_{r+1} &= \sum_{j=0}^{u2^r-1} b(3;j) = \sum_{j=0}^{t2^r-1}\{b(3;2j)+b(3;2j+1)\} \\ &= \sum_{j=0}^{t2^r-1}\{b(3;j)+b(3;j-1)+b(3;j)\} = 3A_r - b(3;t\cdot 2^r-1). \end{aligned} \tag{5.19}$$

Since $b(3;t\cdot 2^r-1) = s(t\cdot 2^r) = s(2t) = b(3;u-1)$, (5.19) implies that

$$A_{r+1} - \frac{1}{2}b(3;u-1) = 3\{A_r - \frac{1}{2}b(3;u-1)\}, \tag{5.20}$$

from which (5.18) is immediate. ∎

This theorem has interesting consequences for the behavior of

$$\hat{b}(6;n) = b(6;n)n^{-\lambda(6)} = b(6;n)n^{-\log_2 3}. \tag{5.21}$$

Proposition 5.22 (Carlitz). $\lim_{n\to\infty}\hat{b}(6;n)$ *does not exist.*

Proof: We apply Thm. 5.19 with $u = 2$ and $u = 6$, obtaining:

$$b(6;2\cdot 2^r-1) = \frac{1}{2}(3^r+1), \tag{5.23(i)}$$

$$b(6;6\cdot 2^r-1) = 3^{r+1}+1. \tag{5.23(ii)}$$

Thus,

$$\hat{b}(6;2\cdot 2^r-1) = \frac{1}{2}(3^r+1)(2^{r+1}-1)^{-\log_2 3} \to 1/6 \cong .1667 \tag{5.24(i)}$$

$$\hat{b}(6;6\cdot 2^r-1) = (3^{r+1}+1)(3\cdot 2^{r+1}-1)^{-\log_2 3} \to 3^{-\log_2 3} \cong .1753. \ ∎ \tag{5.24(ii)}$$

Carlitz writes in [**C3**,p.151]: "P. T. Bateman has suggested that it would be of interest to examine the sum function

$$S(n) = \sum_{k=0}^{n}\theta_0(k)." \tag{5.25}$$

We have seen that $S(n) = b(6;2n-2)$. The computations (5.24) appear, in effect, in [**C3**,p.152].

It can be shown that there exists a continuous function ψ on $[1,2]$ so that, if $u = m/2^k$ is a dyadic rational in $[1,2]$, then $\hat{b}(6;u2^r) \to \psi(u)$. A proof of this result will appear elsewhere [**R3**].

6. Open Questions and Acknowledgements

We believe there is still much to learn about binary partition functions, let alone their analogues for bases other than 2.

What other recurrences are satisfied by binary partition functions? How can the various properties of $b(\infty;n)$ be regarded as the limit of properties of finite $b(d;n)$'s? For example, Knuth remarks that

$$b(\infty;4n)^2 - b(\infty;4n-2)b(\infty;4n+2) = b(\infty;2n)^2 \tag{6.1}$$

is an immediate consequence of (2.9); similarly, by Thm. 2.4(vii),

$$b(4d;4n)^2 - b(4d;4n-2)b(4d;4n+2) = b(2d;2n)^2. \tag{6.2}$$

There do not seem to be easy generalizations of Thm. 2.14 to moduli greater than 2; whenever the answer is known, the set

$$\mathcal{A}(d;m,a) = \{n : b(d;n) \equiv a \pmod{m}\} \qquad (6.3)$$

is either finite or has a well-defined positive density. This is clear for $m = 2$. Since $f(r,t)(s)$ is an integer valued polynomial of degree $r-1$, it is easy to show that $f(r,t)(s) \pmod{m}$ is periodic for all s and m. It follows that $\mathcal{A}(2^r;m,a)$ is a finite union of disjoint arithmetic progressions. In **[R2]** we compute the density of $\mathcal{A}(3;m,a)$, which is determined by the primes which divide m and a. For example, if p is prime, $\mathcal{A}(3;p,0)$ has density $1/(p+1)$ and, if $1 \le a \le p-1$, then $\mathcal{A}(3;p,a)$ has density $p/(p^2-1)$. Churchhouse's results on $b(\infty;n) \pmod 4$ also imply that $\mathcal{A}(\infty;4,0)$ has density $1/3$ and $\mathcal{A}(\infty;4,2)$ has density $2/3$. We risk a conjecture.

Conjecture 6.4. *For all d, a and m, $\mathcal{A}(d;m,a)$ has well-defined density $\alpha = \alpha(d,m,a)$.*

The asymptotic analysis of $b(d;n)$ in section four begs a number of questions. What are the values of (or estimates for)

$$\alpha(2k) = \varliminf_{n\to\infty} b(k;n)n^{-\log_2 k}, \qquad (6.5)(i)$$

$$\beta(2k) = \varlimsup_{n\to\infty} b(k;n)n^{-\log_2 k}; \qquad (6.5)(ii)$$

are these ever equal, except when $k = 2^r$? What are the actual values of

$$\lambda_1(2k+1) = \varlimsup_{n\to\infty} \log(b(2k+1;n))/(\log n), \qquad (6.6)(i)$$

$$\lambda_2(2k+1) = \varliminf_{n\to\infty} \log(b(2k+1;n)/(\log n)? \qquad (6.6)(ii)$$

Is $c \cdot n^\lambda$ the "correct" bound; that is, is it true that, for all $k \ge 1$,

$$\infty > \varlimsup_{n\to\infty} b(2k+1;n)n^{-\lambda_1(2k+1)}, \qquad (6.7)(i)$$

$$\varliminf_{n\to\infty} b(2k+1;n)n^{-\lambda_2(2k+1)} > 0? \qquad (6.7)(ii)$$

Are there any Churchhouse-like formulas (viz. (1.3)) for $b(d;n)$'s, $d < \infty$? How does $b(d;n)/b(\infty;n)$ behave; for which d_n is $b(d_n;n) \sim b(\infty;n)/2$? Are there more combinatorial interpretations for the recurrences?

We thank Paul Bateman for providing a supportive atmosphere at the University of Illinois for number theory, without which this paper would never have been attempted. We also thank the organizers of the conference for their efforts and the editors of this collection, for their generosity in considering this manuscript.

7. Appendix

Here is a table of $b(d;n)$ for $2 \le d \le 9$ and $d = \infty$, and $0 \le n \le 32$.

$n\backslash d$	2	3	4	5	6	7	8	9	∞
0	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2
3	1	1	2	2	2	2	2	2	2
4	1	3	3	4	4	4	4	4	4
5	1	2	3	3	4	4	4	4	4
6	1	3	4	5	5	6	6	6	6
7	1	1	4	4	5	5	6	6	6
8	1	4	5	8	8	9	9	10	10
9	1	3	5	6	8	8	9	9	10
10	1	5	6	9	10	12	12	13	14
11	1	2	6	7	10	10	12	12	14
12	1	5	7	12	13	16	16	18	20
13	1	3	7	8	13	14	16	16	20
14	1	4	8	12	14	19	20	22	26
15	1	1	8	9	14	15	20	20	26
16	1	5	9	17	18	24	25	30	36
17	1	4	9	12	18	20	25	26	36
18	1	7	10	18	21	28	30	35	46
19	1	3	10	14	21	22	30	31	46
20	1	8	11	23	26	34	36	44	60
21	1	5	11	15	26	29	36	38	60
22	1	7	12	22	28	39	42	50	74
23	1	2	12	16	28	30	42	44	74
24	1	7	13	28	33	46	49	62	94
25	1	5	13	19	33	38	49	52	94
26	1	8	14	27	36	52	56	68	114
27	1	3	14	20	36	40	56	59	114
28	1	7	15	32	40	59	64	81	140
29	1	4	15	20	40	49	64	68	140
30	1	5	16	29	41	64	72	88	166
31	1	1	16	21	41	48	72	76	166
32	1	6	17	38	46	72	81	106	202

Table 7.1

REFERENCES

[A1] Andrews, G., Congruence properties of the m-ary partition function, J. Number Theory, **3** (1971), 104-110. MR 42#3043.

[A2] Andrews, G., The Theory of Partitions. Encyc. of Math. and Appl., v.2, Addison-Wesley, Reading 1976, MR 58#27738.

[B] de Bruijn, N. G., On Mahler's partition problem, Indag. Math. **10** (1948), 210-220. MR 10, 16d.

[C1] Carlitz, L., A problem in partitions related to the Stirling numbers, Riv. Mat. Univ. Parma **5** (1964), 61-75. MR 34#158.

[C2] Carlitz, L., A problem in partitions related to the Stirling numbers, Bull. Amer. Math. Soc. **70** (1964), 275-278. MR 28#1135.

[C3] Carlitz, L., Generating functions and partition problems. Pp. 144-169, in Proc. Sympos. Pure Math. VIII, Amer. Math. Soc., Providence, 1965, MR 31 #72.

[C4] Churchhouse, R. F., Congruence properties of the binary partition function, Proc. Camb. Phil. Soc. **66** (1969), 371-376. MR 40#1356.

[D1] Dirdal, G., On restricted m-ary partitions, Math. Scand. **37** (1975), 51-60. MR 52#10582.

[D2] Dirdal, G., Congruences for m-ary partitions, Math. Scand. **37** (1975), 76-82. MR 52#10583.

[E1] Euler, L., Introductio in analysin infinitorum. Lausanne, 1748, in Opera Omnia Series Prima Opera Math. vol. 8, B. G. Teubner, Leipzig, 1922.

[E2] Euler, L., De partitione numerorum, Nov. com. acad. sci. Petro. **3** (1750/1751), 125-169. In Opera Omnia Series Prima Opera Math. vol. 2, pp. 254-294, B. G. Teubner, Leipzig, 1915.

[G1] Gupta, H., Proof of the Churchhouse conjecture concerning binary partitions, Proc. Camb. Phil. Soc. **70** (1971), 53-56. MR 45#4986.

[G2] Gupta, H., A simple proof of the Churchhouse conjecture concerning binary partitions, Ind. J. Pure Appl. Math. **3** (1972), 791-794. MR 48#8377.

[G3] Gupta, H., On m-ary partitions, Proc. Camb. Phil. Soc **71** (1972), 343-345. MR 45#204.

[G4] Gupta, H., A direct proof of the Churchhouse conjecture concerning binary partitions, Ind. J. Math. **18** (1976), 1-5. MR 82b:10013.

[GP] Gupta, H. and P. A. B. Pleasants, Partitions into powers of m, Ind. J. Pure Appl. Math. **10** (1979), 655-694. MR 80f:10014.

[HL] Hirschhorn, M. D. and J. H. Loxton, Congruence properties of the binary partition function, Proc. Camb. Phil. Soc. **78** (1975), 437-442. MR 52#3045.

[KAH] Klosinski, L. F., G. L. Alexanderson and A. P. Hillman, The William Lowell Putnam Mathematical Competition, Amer. Math. Monthly **91** (1984), 487- 495.

[K] Knuth, D. E., An almost linear recurrence, Fib. Q. **4** (1966), 117-128. MR 33#7317.

[L1] Lehmer, D. H., On Stern's diatomic series, Amer. Math. Monthly **36** (1929), 59-67.

[L2] Lind, D. A., An extension of Stern's diatomic series, Duke Math. J. **36** (1969), 55-60. MR 30 #6810.

[L3] Lucas, É., Sur les suites de Farey, Bull. Soc. Math. France (1877 -1878), 118-119.

[M] Mahler, K., On a special functional equation, J. London Math. Soc. **15** (1940), 115-123. MR 2, 133e.

[R1] Reznick, B., Digital representations using the greatest integer function, Trans. Amer. Math. Soc. **312** (1989), 355-375. MR 89g:11010.

[R2] Reznick, B., Congruence properties of the Stern sequence. in preparation.

[R3] Reznick, B., Stern measures. in preparation.

[R4] de Rham, G., Un peu de mathématiques à propos d'une courbe plane, Elem. Mat. **2** (1947), 73-76, 89-97. MR 9-246.

[R5] Rödseth, O., Some arithmetical properties of m-ary partitions, Proc. Camb. Phil. Soc. **68** (1970), 447-453. MR 41#5319.

[S1] Sloane, N. J. A., A Handbook of Integer Sequences, Academic Press, New York (1973). MR 50#9760.

[S2] Stanley, R. P. Private communication.

[S3] Stern, M. A., Ueber eine zahlentheoretische funktion, J. für Math. **55** (1858), 193-220.

[T1] Tanturri, A, Della partizione dei numeri. Ambi, terni quaterne e cinquine di data somma, Att. delle Sci. di Torino **52** (1916/7), 902-918.

[T2] Tanturri, A, Sul numero delle partizioni d'un numero in potenze di 2, Att. delle Sci. di Torino **54** (1918), 97-110.

[T3] Tanturri, A., Sul numero delle partizioni d'un numero in potenze di 2, Att. Accad. Naz. Lincei **27** (1918), 399-403.

Bruce Reznick
Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, Illinois 61801