CONTRIBUTIONS TO THE PROOF THEORY OF HYPERGEOMETRIC IDENTITIES

A Dissertation in Mathematics

Presented to the Faculties of the University of Pennsylvania
in Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy

1993

Lily Yen

Herbert S. Wilf
Supervisor

Ted Chinburg
Graduate Group Chairman

To my parents

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Herbert S. Wilf, for his encouragement and support, especially for sharing with me his enthusiasm for mathematical research.

To my husband, Mogens Lemvig Hansen, I owe many special thanks for numerous mathematical conversations, for teaching me how to TeX, being my personal TeXnical support, for typing some of my thesis, and for bravely and persistently criticizing the exposition.

This dissertation was typeset by AMS-TeX, the TeX macro system of the American Mathematical Society.

ABSTRACT

CONTRIBUTIONS TO THE PROOF THEORY OF HYPERGEOMETRIC IDENTITIES

LILY YEN

Herbert S. Wilf

In 1992 Wilf and Zeilberger introduced the following terminology: A *hypergeometric term* is a function $F(k_1, k_2, \ldots, k_r)$ such that, for all $i \in \{1, 2, \ldots, r\}$, the ratio

$$\frac{F(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_r)}{F(k_1, \ldots, k_r)}$$

is a rational function in all the variables. They also introduced the rather technical concept of *admissible proper*-hypergeometric terms; "most interesting" hypergeometric terms are admissible and proper.

We prove the following: Given an integer n_0 and an admissible proper-hypergeometric term $F(n, k)$, there exists a pre-computable integer n_1 such that if $\sum_k F(n, k) = 1$ for $n_0 \le n \le n_1$, then $\sum_k F(n, k) = 1$ for *all* $n \ge n_0$. Moreover, an a priori upper bound is given for n_1. This allows us to prove many hypergeometric identities by simply checking a finite (albeit large) number of initial values. With similar methods, we show explicit a priori upper bounds for n_1 in the cases where $\sum_k F(n, k) = f(n)$ (for some hypergeometric term $f(n)$) and $\sum_k F(n, k) = \sum_k G(n, k)$ (for some admissible proper-hypergeometric term $G(n, k)$) are the objects of interest. Finally, we generalize the above statement to the case of $\sum_{k_1} \sum_{k_2} \cdots \sum_{k_r} F(n, k_1, k_2, \ldots, k_r) = 1$.

CONTENTS

INTRODUCTION

The study of ordinary and partial differential equations led to the investigation of special functions, those bearing the names of Gauss, Hermite, Jacobi, Laguerre and Legendre. Therefore, Askey [As3] defined special functions as "functions that occur often enough to merit a name". Most special functions are expressible as hypergeometric series, i.e. a series $\sum_{k=0}^{\infty} a_k$ such that the ratio a_{k+1}/a_k of consecutive terms is a rational function of k. For example, the Hermite polynomials

$$\text{(Hermite)} \qquad H_n(x) := n! \sum_k \frac{(-1)^k (2x)^{n-2k}}{(n-2k)!k!}$$

has $a_{k+1}/a_k = -(n-2k)(n-2k-1)/(4x^2(k+1))$; the Laguerre polynomials

$$\text{(Laguerre)} \qquad L_n^{\alpha}(x) := \sum_k \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}$$

has $a_{k+1}/a_k = (n-k)(-x)/((\alpha+k+1)(k+1))$; the Legendre polynomials

$$\text{(Legendre)} \qquad P_n(x) := \frac{1}{2^n} \sum_k \binom{n}{k}^2 (x-1)^k (x+1)^{n-k}$$

has $a_{k+1}/a_k = (n-k)^2(x-1)/((x+1)(k+1)^2)$; and the general Jacobi polynomials

$$\text{(Jacobi)} \qquad P_n^{(\alpha,\beta)}(x) := \frac{1}{2^n} \sum_k \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}$$

has

$$\frac{a_{k+1}}{a_k} = \frac{(x+1)(n+\alpha+1)(n+\beta+1)}{(n-k+1)(n+\beta-k+1)}.$$

The first hypergeometric series that rose to fame and became *the* hypergeometric series of the 19th century was the ${}_2F_1$, often called the Gaussian hypergeometric, for Gauss in his doctoral dissertation of 1812 [Gau] presented a thorough investigation of the series. Prior to Gauss, Euler [E] and Pfaff [Pf] also discovered many remarkable properties of ${}_2F_1$. The study of hypergeometric series became so important that W. W. Sawyer once remarked [S] "There must be many universities today where 95 per cent, if not 100 per cent, of the functions studied by physics, engineering, and even mathematics students, are covered by this single symbol $F(a,b;c;x)$ [i. e. ${}_2F_1$]."

In 1870, ${}_2F_1$ was generalized to ${}_mF_n$.

Definition. [GKP, p. 205] The *general hypergeometric series* is a power series in z with $m+n$ parameters, and it is defined as follows in terms of rising factorial powers:

$${}_mF_n\left[\begin{matrix} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{matrix}\middle| z\right] = \sum_{k\geq 0} \frac{a_1^{\overline{k}}\ldots a_m^{\overline{k}}}{b_1^{\overline{k}}\ldots b_n^{\overline{k}}}\frac{z^k}{k!},$$

where $a^{\overline{k}}$ (also denoted by $(a)_k) := a(a+1)(a+2)\ldots(a+k-1)$. To avoid division by zero, none of the b's may be zero or a negative integer. Other than that, the a's and b's may be anything. The a's are said to be *upper parameters*, and the b's are *lower parameters*. The last quantity z is called the *argument*.

We should note that most literature about hypergeometric series uses the notation in the definition. Sometimes, a one-line notation '$F(a_1,\ldots,a_m;b_1,\ldots,b_n;z)$' is also used (as in Sawyer's remark). However, Graham, Knuth and Patashnik do not have subscripts m and n around F in [GKP] because it is clear how many parameters are upper and lower parameters.

We are now witnessing a fast comeback of special functions and their associated hypergeometric series. Moreover, the q-analogues of special functions and hypergeometric series, called q-series have proved to be very useful in number theory, combinatorics, physics, group theory, [An5] and other areas of science and mathematics.

Andrews in 1974 [An1] first pointed out the great relevance of hypergeometric series to binomial coefficient identities. Indeed, special functions and hypergeometric series satisfy many identities, most of which involve binomial coefficients. We quote the following paragraph from [WZ4, p. 148 ¶2].

> There are countless identities relating special functions (e.g., [PBM, R, An5, As1]. In addition to their intrinsic interest, some of them imply important properties of these special functions, which in turn sometimes imply deep theorems elsewhere in mathematics (e.g., [deB, Ap]). Just as important for mathematics are the extremely successful attempts to instill meaning and insight, both representation-theoretic (e.g., [Mi]) and combinatorial (e.g., [Fo2]), into these identities.

Special functions share an even more remarkable property recently pointed out in [Z2, Z4, WZ2]: Most special functions can be written in the form

$$P_n = \sum_{k=0}^{\infty} F(n,k)$$

where n is an auxiliary parameter, and one has that not only is $F(n,k+1)/F(n,k)$ a rational function of k, but is a rational function of (n,k), and in addition, so is $F(n+1,k)/F(n,k)$. It is easy to check that $F(n+1,k)/F(n,k)$ is indeed a rational function of (n,k)

in the examples given before. We will call such an F a *hypergeometric term*[1] as in [WZ3]. This observation led Zeilberger [Z4] to conclude that a hypergeometric term is [WZ3] "an entirely rational, finitary object," and "can be handled by finite methods and machines [Z4], [WZ1], [WZ2]." Thus was born Wilf and Zeilberger's algorithmic proof theory for hypergeometric identities [WZ3].

Sister Celine Fasenmyer working under the supervision of Rainville found an algorithm for obtaining recurrence relations satisfied by hypergeometric polynomials. She presented the method by examples in her Ph. D. thesis [F1] in 1945 and in two subsequent papers [F2, F3]. Before the 1940's, 'it seemed customary upon entering the study of a new set of polynomials to seek recurrence relations, pure or mixed by essentially a hit-and-miss process' [R, p. 233]. With Sister Celine's technique, it was possible to find pure recurrences for a certain class of hypergeometric polynomials. Verbaeten [V] in 1974 showed how to make her technique general in the one summation case.

Independent of Verbaeten's work, Zeilberger [Z2] showed how to apply Sister Celine's method systematically. Furthermore, Zeilberger realized that Sister Celine's technique implies *all binomial identities are trivial* in the sense that one only needs to check a finite number of special cases to establish the truth of the identity of interest. Indeed, Zeilberger is the first to realize that Sister Celine's technique opened the door to automatic proving of hypergeometric identities. Central to Zeilberger's discovery is the fact that given a proposed hypergeometric expression $\sum_k F(n,k) = \sum_k G(n,k)$, we can show that the equality holds for all n by showing that both $\sum_k F(n,k)$ and $\sum_k G(n,k)$ satisfy the same

[1] More generally, a function $F(k_1, k_2, \ldots, k_r)$ is said to be a *hypergeometric term* if, for all $i \in [r]$, the ratio $F(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_r)/F(k_1, \ldots, k_r)$ is a rational function in all the variables.

recurrence and agree for some initial values of n.

Zeilberger's development of the proof theory for hypergeometric multisum identities began in the late 70's. A decade later, Wilf and Zeilberger employed Gosper's algorithm [G] in the discovery of WZ-pairs for proving hypergeometric identities [WZ1, WZ2, Z5]. (Almost all known single-sum hypergeometric identities can be proved using WZ-pairs.) Recently [WZ3], Wilf and Zeilberger formalized, systematized, and generalized Sister Celine' technique to prove hypergeometric identities. They defined *proper*-hypergeometric terms [WZ3, p. 596] for which her method will always produce recurrence relations. For the first time, an explicit a priori upper bound for the order of the recurrence satisfied by the hypergeometric term $F(n, \mathbf{k})$ is known [WZ3, Theorem 3.1]. Further, they gave admissibility conditions [WZ3, p. 602] on $F(n, \mathbf{k})$ for $\sum_{\mathbf{k}} F(n, \mathbf{k})$ to satisfy the same recurrence as $F(n, \mathbf{k})$. In addition to the proof theory for (multisum) hypergeometric identities, they successfully applied Sister Celine's technique to q-hypergeometric identities to obtain an a priori upper bound for the order of the recurrence, and for the first time presented an algorithmic proof theory for the q-hypergeometric identities. Combining the notion of WZ-pairs and the proof theory for multisum ordinary/q hypergeometric identities, they showed how to prove ordinary/q hypergeometric identities using WZ-tuples. (Again, almost all known identities satisfy recurrence relations in the form of WZ-tuples.) The proof theory was also extended to identities involving multiple integrals. For this dissertation, we will consider Wilf–Zeilberger's algorithmic proof theory only for the discrete ordinary single/multisum identities.

Sister Celine Fasenmyer in her Ph. D. dissertation [F1] presented many examples of hypergeometric series $\sum_k F(n, k)x^k$ for which she found recurrence relations by first obtaining the recurrence for $F(n,k)x^k$. Her technique finds a recurrence relation for the hypergeometric term, $F(n,k)x^k$ with polynomial-in-(n,x) coefficients. Three decades later, Zeilberger applied Sister Celine's method for proving proposed hypergeometric identities [Z1, Z2] in the following way. Suppose we would like to show that $\sum_k F(n,k) = f(n)$, where $F(n,k)$ and $f(n)$ are hypergeometric terms. Then we consider the ordinary generating function of $F(n,k)$, namely, $\sum_k F(n,k)x^k$, and obtain a recurrence relation for $F(n,k)x^k$ using Sister Celine's method. After dividing the recurrence relation by the smallest common factor $(x-1)^l$, and setting $x = 1$, we get a recurrence for $F(n,k)$. If we sum over k, we will, if lucky, get a recurrence for the sum $\sum_k F(n,k)$. Because the coefficients of the recurrence relation are polynomials in (n,x) by Sister Celine's technique, the coefficients of the recurrence for the sum $\sum_k F(n,k)$ are polynomials in n only. It is now trivial to check whether $f(n)$ satisfies this recurrence relation. If this is so, and if $f(n) = \sum_k F(n,k)$ for certain initial values of n, then it follows by induction, that $f(n) = \sum_k F(n,k)$ for all n. The necessary initial values to check are the numbers up to (and including) the sum of the order of the recurrence and the highest integer zero of the leading (polynomial-in-n) coefficient of the recurrence. In short, *we have reduced proving the identity into checking a few initial values of* n. Furthermore, Zeilberger expressed the view [Z2, p. 122] that given $\sum_k F(n,k) = f(n)$, where $F(n,k)$ and $f(n)$ are hypergeometric terms, there exists an n_1 such that the identity $\sum_k F(n,k) = f(n)$ is true for all n if (and only if) it is true for $n \leq n_1$. We give an explicit, pre-computable n_1 in this paper. (See Theorem 5.1 and its proof in Chapter 5.)

In Chapter 1, we follow the proof of [WZ3, Theorem 3.1] and sharpen upper bounds for the order of the recurrence satisfied by the summand in the case of just one summation index.

Chapter 2 contains an algorithm for finding the certificate $R(n,k)$, a rational function in n and k, needed to prove identities in the WZ-pair fashion. The algorithm is similar to the one described in [WZ3, pp. 592–593] for finding the certificate $R(n,k)$ directly. It uses the sharper upper bounds from Chapter 1.

Chapter 3 is a multivariable version of Chapter 1. To accomplish this generalization, we need to solve a certain minimization problem, estimate the number of positive zeros of a particular polynomial, and find an upper bound for the zeros of that polynomial.

Chapter 4 is the multivariable analogue of Chapter 2. We present an algorithm for finding the certificates $R_i(n,\mathbf{k})$ for $i \in [r]$ that are needed in proving identities using WZ-tuples. As in Chapter 2 which used bounds from Chapter 1, the bounds from Chapter 3 are used in Chapter 4.

In response to [WZ3, §2.3, end of ¶2], we show in Chapter 5 some examples of hypergeometric sums whose recurrence have leading coefficients that vanish at positive integers where the sums are valid. We devote most of the chapter to the proof—using results from Chapters 1 and 2—of our

Main Theorem. *Let*

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k$$

be an admissible proper-hypergeometric term, and $P(n,k)$ be a polynomial with coefficients *in $\mathbb{Z}$. Let*

$$x := \max_s\{|a_s|, |b_s|, |c_s|, |u_s|, |v_s|, |w_s|\},$$

$$y := \max\{p, q\},$$

$$z := \max_{0\le i,j} \left|[n^j k^i]P(n,k)\right|,$$

$$d := 1 + \max\{\deg_k P(n,k), \deg_n P(n,k)\},$$

and let n_0 be a given integer. If $\sum_k F(n,k) = 1$ for

$$n_0 \le n \le (3xy)^{3(d+1)^2(2xy)^6} d^{5(d+1)(2xy)^3} z^{(d+1)(2xy)^3},$$

then $\sum_k F(n,k) = 1$ for <u>all</u> $n \ge n_0$.

In the last section of Chapter 5 we generalize the Main Theorem to the cases where the equations $\sum_k F(n,k) = f(n)$ (for some hypergeometric term $f(n)$) and $\sum_k F(n,k) = \sum_k G(n,k)$ (for some admissible proper-hypergeometric term) are the objects of interest.

We generalize Theorem 5.1 to multiple summation indices in Chapter 6.

Chapter 7 contains a sufficient condition on $F(n,k)$ for the sum, $\sum_k F(n,k)$, to be hypergeometric—or equivalently, to be summable in closed form. The sum $\sum_k F(n,k) =: f(n)$ is hypergeometric if $f(n)/f(n+1) = P(n)/Q(n)$ for some polynomials, P and Q, in n. Notice that in this case, $P(n)f(n+1) - Q(n)f(n) = 0$, so $f(n)$ is a solution to a first order recurrence relation (in n) with polynomial-in-n coefficients.

Petkovšek, in his Ph. D. dissertation [P], gives an algorithm that solves the following decision problem:

Given a linear recurrence relation of order h with polynomial coefficients, decide whether the recurrence has a solution that satisfies another recurrence of order 1; and if so, find that recurrence of order 1.

In other words, Petkovšek gives necessary conditions on the polynomial coefficients of the recurrence for the existence of a hypergeometric solution to the recurrence. Petkovšek's algorithm works only if the recurrence contains no free parameters. We still do not know any necessary condition on an admissible proper-hypergeometric term, $F(n,k)$, for the sum $\sum_k F(n,k)$ to be hypergeometric.

CHAPTER I

THE ORDER OF THE RECURRENCE FOR $F(n,k)$

We show slightly better upper bounds for the order of the recurrence satisfied by a given proper-hypergeometric term $F(n,k)$. We follow the proof of Theorem 3.1 in [WZ3] and hold fast unto the estimates to obtain our bounds.

Definition 1.1. [WZ3] A *proper-hypergeometric term* is a function of the form

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k, \tag{1.1}$$

where P is a polynomial and ξ is a parameter. The a's, b's, u's and v's are assumed to be specific integers, i.e., they are integers and do not depend on any other parameters. The c's and the w's are also integers, but they may depend on parameters. We will say that F is *well-defined* at (n,k) if none of the numbers $\{a_s n + b_s k + c_s\}_1^p$ is a negative integer. We will say that $F(n,k) = 0$ if F is well-defined at (n,k) and at least one of the numbers $\{u_s n + v_s k + w_s\}_1^q$ is a negative integer, or $P(n,k) = 0$.

Definition 1.2. [WZ3] A proper-hypergeometric term F is said to satisfy a k-free recurrence at a point $(n_0, k_0) \in \mathbb{Z}^2$ if there are integers I, J and polynomials $\alpha_{i,j} = \alpha_{i,j}(n)$ that do not depend on k and are not all zero, such that the relation

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}(n)F(n-j, k-i) = 0 \tag{1.2}$$

holds for all (n,k) in some $\mathbb{R}^2$ neighborhood of (n_0,k_0), in the sense that F is well-defined at all of the arguments that occur, and the relation (1.2) is true.

Theorem 1.3. [WZ3, Theorem 3.1] *Every proper-hypergeometric term F satisfies a non-trivial k-free recurrence relation. Indeed there exist I,J and polynomials $\alpha_{i,j}(n)$ ($i = 0,\dots,I; j = 0,\dots,J$) not all zero, such that* (1.2) *holds at every point $(n_0,k_0) \in \mathbb{Z}^2$ for which $F(n_0,k_0) \neq 0$ and all of the values $F(n_0-j,k_0-i)$ that occur in* (1.2) *are well-defined. Furthermore there exists such a recurrence with $(I,J) = (I^*,J^*)$, where*

$$J^* = \sum_s |b_s| + \sum_s |v_s|, \quad I^* = 1 + \deg(P) + J^*\Big(\Big(\sum_s |a_s| + \sum_s |u_s|\Big) - 1\Big). \tag{1.3}$$

1.1 Slightly better upper bounds

Notation. We let $x^+ := \max\{0,x\}$. The set $\{1,2,\dots,I\}$ is denoted by $[I]$, and $[I]_0$ means $[I] \cup \{0\}$. We let $x^{\underline{m}}$ denote $x(x-1)\cdots(x-m+1)$, and $x^{\overline{m}}$ denote $x(x+1)\cdots(x+m-1)$ for positive integers m. We define $x^{\underline{0}} = 1 = x^{\overline{0}}$.

We improve the bounds for I^* and J^* by

Theorem 1.4. *Let*

$$U := \sum_{\substack{s \\ v_s \neq 0}} u_s, \quad V := \sum_s v_s, \qquad A := \sum_{\substack{s \\ b_s \neq 0}} a_s, \quad B := \sum_s b_s,$$
$$\mathcal{A} := \sum_{\substack{s \\ b_s \neq 0}} (a_s)^+ + \sum_{\substack{s \\ v_s \neq 0}} (-u_s)^+, \quad \mathcal{B} := \sum_s (b_s)^+ + \sum_s (-v_s)^+,$$

and $\delta = \deg_k P(n,k)$. Then J^* and I^* in (1.3) *of Theorem 1.3 can be replaced by*

$$J^* = \mathcal{B} + (V-B)^+, \quad \text{and} \quad I^* = 1 + \delta + J^*\left(\mathcal{A} + (U-A)^+ - 1\right).$$

Proof. Fix some I, $J > 0$, and suppose (n_0,k_0) is a point that satisfies the two conditions of the theorem. Since we assumed that all of the a_s, b_s, u_s, v_s in Definition 1.1 are integers, we have that for all (n, k) in some $\mathbb{R}^2$ neighborhood of (n_0, k_0), all of the ratios $F(n-j, k-i)/F(n,k)$ are well-defined *rational functions* of n and k. (See (1.1) for $F(n,k)$.) Hence we can form a linear combination

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}(n)\frac{F(n-j,k-i)}{F(n,k)} \tag{1.4}$$

of these rational functions, in which the α's are to be determined, if possible, so as to make the sum vanish identically in the neighborhood.

As in [WZ3], the problem is to find a common denominator for the summand in (1.4). Instead we find a common denominator $D(n,k)$ for

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\frac{F(n-j,k-i)}{F(n,k)}.$$

Clearly, $D(n,k)$ is also a common denominator for the summand in (1.4).

Consider

$$\frac{F(n-j,k)}{F(n,k)} = \frac{P(n-j,k)}{P(n,k)}\prod_{s=1}^{p}\frac{(a_s n + b_s k + c_s - a_s j)!}{(a_s n + b_s k + c_s)!}\prod_{s=1}^{q}\frac{(u_s n + v_s k + w_s)!}{(u_s n + v_s k + w_s - u_s j)!} \tag{1.5}$$

which contributes to the denominator $D(n,k)$, if $a_s > 0$, or $u_s < 0$, or both.

In (1.5), if $a_s > 0$ for some $s \in [p]$, then

$$\frac{(a_s n + b_s k + c_s - a_s j)!}{(a_s n + b_s k + c_s)!} = \frac{1}{(a_s n + b_s k + c_s)^{\underline{a_s j}}}.$$

Since $(a_s n + b_s k + c_s)^{\underline{a_s j}}$ divides $(a_s n + b_s k + c_s)^{\underline{a_s J}}$ for $0 < j \le J$ and $a_s > 0$, a common denominator for $\sum_{j=0}^{J}\frac{F(n-j,k)}{F(n,k)}$ is

$$P(n,k)\prod_{\substack{s=1\\ a_s>0}}^{p}(a_s n + b_s k + c_s)^{\underline{a_s J}}\prod_{\substack{s=1\\ u_s<0}}^{q}(u_s n + v_s k + w_s + 1)^{\overline{-u_s J}}. \tag{1.6}$$

Similarly, a common denominator for $\sum_{i=0}^{I} \frac{F(n,k-i)}{F(n,k)}$ is

$$P(n,k) \prod_{\substack{s=1 \\ b_s>0}}^{p} (a_s n + b_s k + c_s)^{\overline{b_s I}} \prod_{\substack{s=1 \\ v_s<0}}^{q} (u_s n + v_s k + w_s + 1)^{\overline{-v_s I}}. \tag{1.7}$$

Putting (1.6) and (1.7) together, we have

$$D(n,k) = P(n,k) \prod_{s=1}^{p} (a_s n + b_s k + c_s)^{\overline{\max_{i,j}(a_s j + b_s i)^+}} \prod_{s=1}^{q} (u_s n + v_s k + w_s + 1)^{\overline{\max_{i,j}(-u_s j - v_s i)^+}}.$$

Clearly,

$$\max_{\substack{i\in[I]_0 \\ j\in[J]_0}} (a_s j + b_s i)^+ = (a_s)^+ J + (b_s)^+ I,$$

and

$$\max_{\substack{i\in[I]_0 \\ j\in[J]_0}} (-u_s j - v_s i)^+ = (-u_s)^+ J + (-v_s)^+ I.$$

If we let $\delta := \deg_k P(n,k)$, then the degree in k of $D(n,k)$ is

$$\begin{aligned}
&\delta + J\Big(\sum_{\substack{s\in[p] \\ b_s\neq 0}} (a_s)^+\Big) + I\Big(\sum_{s\in[p]} (b_s)^+\Big) + J\Big(\sum_{\substack{s\in[q] \\ v_s\neq 0}} (-u_s)^+\Big) + I\Big(\sum_{s\in[q]} (-v_s)^+\Big) \\
&= \delta + J\Big(\sum_{\substack{s\in[p] \\ b_s\neq 0}} (a_s)^+ + \sum_{\substack{s\in[q] \\ v_s\neq 0}} (-u_s)^+\Big) + I\Big(\sum_{s\in[p]} (b_s)^+ + \sum_{s\in[q]} (-v_s)^+\Big).
\end{aligned}$$

Next, we find the degree in k of the numerator polynomial $N(n,k)$ in (1.4) with $D(n,k)$ as the common denominator. Consider the (i,j)th term in

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \frac{F(n-j,k-i)}{F(n,k)}. \tag{1.8}$$

Since

$$\frac{F(n-j,k-i)}{F(n,k)} = \frac{P(n-j,k-i)}{P(n,k)} \xi^{-i}$$
$$\times \frac{\displaystyle\prod_{\substack{s=1 \\ a_s j + b_s i<0}}^{p} (a_s n + b_s k + c_s + 1)^{\overline{-a_s j - b_s i}} \prod_{\substack{s=1 \\ u_s j + v_s i>0}}^{q} (u_s n + v_s k + w_s)^{\overline{u_s j + v_s i}}}{\displaystyle\prod_{\substack{s=1 \\ a_s j + b_s i>0}}^{p} (a_s n + b_s k + c_s)^{\overline{a_s j + b_s i}} \prod_{\substack{s=1 \\ u_s j + v_s i<0}}^{q} (u_s n + v_s k + w_s + 1)^{\overline{-u_s j - v_s i}}},$$

by letting

$$N_{i,j} := \prod_{\substack{s=1 \\ a_s j + b_s i < 0}}^{p} (a_s n + b_s k + c_s + 1)^{\overline{-a_s j - b_s i}} \prod_{\substack{s=1 \\ u_s j + v_s i > 0}}^{q} (u_s n + v_s k + w_s)^{\underline{u_s j + v_s i}},$$

and

$$D_{i,j} := \prod_{\substack{s=1 \\ a_s j + b_s i > 0}}^{p} (a_s n + b_s k + c_s)^{\underline{a_s j + b_s i}} \prod_{\substack{s=1 \\ u_s j + v_s i < 0}}^{q} (u_s n + v_s k + w_s + 1)^{\overline{-u_s j - v_s i}},$$

we have

$$\frac{F(n-j,k-i)}{F(n,k)} = \frac{P(n-j,k-i)}{P(n,k)} \xi^{-i} \frac{N_{i,j} D(n,k)}{D_{i,j} D(n,k)}.$$

Hence, the degree in k of the numerator of the (i,j)th term in (1.8) with $D(n,k)$ as the denominator, i.e., $P(n-j,k-i)\xi^{-i} N_{i,j} D(n,k) / \big(D_{i,j} P(n,k)\big)$, is

$$\begin{aligned} (1.9)\quad \delta + \sum_{\substack{b_s \neq 0 \\ a_s j + b_s i < 0}} (-a_s j - b_s i) + \sum_{\substack{v_s \neq 0 \\ u_s j + v_s i > 0}} (u_s j + v_s i) \\ + \deg_k D(n,k) - \sum_{\substack{b_s \neq 0 \\ a_s j + b_s i > 0}} (a_s j + b_s i) - \sum_{\substack{v_s \neq 0 \\ u_s j + v_s i < 0}} (-u_s j - v_s i) - \delta \\ = \deg_k D(n,k) + \sum_{v_s \neq 0} (u_s j + v_s i) - \sum_{b_s \neq 0} (a_s j + b_s i). \end{aligned}$$

Taking the maximum over i,j of the last line of (1.9) gives

$$\begin{aligned} \deg_k N(n,k) &= \max_{i,j} \Big(\deg_k D(n,k) + \sum_{v_s \neq 0} (u_s j + v_s i) - \sum_{b_s \neq 0} (a_s j + b_s i) \Big) \\ &= \deg_k D(n,k) + \max_{i,j} \Big(j \sum_{\substack{s \\ v_s \neq 0}} u_s + i \sum_{s} v_s - j \sum_{\substack{s \\ b_s \neq 0}} a_s - i \sum_{s} b_s \Big). \end{aligned}$$

Let

$$U := \sum_{\substack{s \\ v_s \neq 0}} u_s, \quad V := \sum_{s} v_s, \quad A := \sum_{\substack{s \\ b_s \neq 0}} a_s, \quad B := \sum_{s} b_s.$$

We can rewrite $\deg_k N(n,k)$ as

$$\begin{aligned}\deg_k N(n,k) &= \deg_k D(n,k) + \max_{i,j}\big(j(U-A) + i(V-B)\big)\\ &= \deg_k D(n,k) + J(U-A)^+ + I(V-B)^+.\end{aligned}$$

Knowing the degree in k of $N(n,k)$, we deduce that there are $1+\deg_k N(n,k)$ homogeneous linear equations to solve in $(I+1)(J+1)$ unknowns, namely, the $\alpha_{i,j}$'s. A system of solutions for the $\alpha_{i,j}$'s exists, if $(I+1)(J+1) \geq 2+\deg_k N(n,k)$. From the inequality, we will obtain an upper bound for J.

Let

$$\mathcal{A} := \sum_{\substack{s\\ b_s\neq 0}} (a_s)^+ + \sum_{\substack{s\\ v_s\neq 0}} (-u_s)^+, \quad \text{and} \quad \mathcal{B} := \sum_s (b_s)^+ + \sum_s (-v_s)^+.$$

Then,

$$\begin{aligned}\deg_k N(n,k) &= \deg_k D(n,k) + \max_{i,j}\Big(j(U-A) + i(V-B)\Big)\\ &= \delta + J\mathcal{A} + I\mathcal{B} + J(U-A)^+ + I(V-B)^+.\end{aligned}$$

If $\mathcal{B} + (V-B)^+ \neq 0$, we let $J^* = \mathcal{B} + (V-B)^+$, and solve for I^* in $(I+1)(J+1) \geq 2+\deg_k N(n,k)$ to get $I^* = 1+\delta+\big(\mathcal{A}+(U-A)^+-1\big)\big(\mathcal{B}+(V-B)^+\big)$ as an upper bound.

If $\mathcal{B} + (V-B)^+ = 0$, namely

$$\sum_s b_s^+ + \sum_s (-v_s)^+ + (\sum_s v_s - \sum_s b_s)^+ = 0,$$

then $b_s = 0$ for all $s \in [p]$, and $v_s = 0$ for all $s \in [q]$. In other words, the factorial part of $F(n,k)$ is independent of k. In this case,

$$\begin{aligned}\sum_k F(n,k) &= \frac{\prod_{s\in[p]}(a_s n + c_s)!}{\prod_{s\in[q]}(u_s n + w_s)!} \sum_k P(n,k)\xi^k\\ &= \frac{\prod_{s\in[p]}(a_s n + c_s)!}{\prod_{s\in[q]}(u_s n + w_s)!} P(n,\xi D)\frac{1}{1-\xi}.\end{aligned}$$

The sum above is summable but infinite. Since we are concerned with only terminating hypergeometric series, we can disregard the case $\mathcal{B} + (V - B)^+ = 0$. □

Remark. If $P(n, k)$ in $F(n, k)$ is a constant, then $\delta = 0$. In this case, the I^* and J^* from Theorem 1.4 agree with the results in [W2] when $\mathcal{B} + (V - B)^+ \neq 0$.

1.2 Examples

Example 1.5. Take $F(n, k) = \binom{n}{k}^2$. We express $F(n, k)$ in the form of Definition 1.1 to get $n!^2/(k!^2(n-k)!^2)$. Then $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, $u_1 = u_2 = 0$, $u_3 = u_4 = 1$, $v_1 = v_2 = 1$, $v_3 = v_4 = -1$, $U = 2$, $V = 0$, $A = 0$, $B = 0$, $\mathcal{A} = 0$, $\mathcal{B} = 2$. Since $U - A = 2$ and $V - B = 0$, we get $J^* = 2$ and $I^* = 3$.

The following two examples are from [W2, p. 4].

Example 1.6. [W2] Fix a positive integer m, and put

$$F(n, k) = \binom{n}{k}^m = \frac{n!^m}{k!^m (n-k)!^m}.$$

Then $a_i = 1$, $i \in [m]$; $b_i = 0$, $i \in [m]$; $u_i = 0$, $i \in [m]$; $u_i = 1$, $i \in [2m] \setminus [m]$; $v_i = 1$, $i \in [m]$; $v_i = -1$, $i \in [2m] \setminus [m]$. Thus $A = 0$, $B = 0$, $U = m$, $V = 0$, $\mathcal{A} = 0$, $\mathcal{B} = m$. Hence $J^* = m$, and $I^* = (m-1)m + 1$.

Example 1.7. [W2] If $F(n, k) = (n+k+\alpha+\beta)!/(k!\,(n-k)!\,(k+\alpha)!)$, then the f_n's where $f_n(x) = \sum_k F(n, k) x^k$ are the Jacobi polynomials. (See Formula (Jacobi) in Introduction for Jacobi polynomials.) A similar calculation as in the previous examples shows that $J^* = 2$ and $I^* = 1$. This is the best possible.

CHAPTER II

AN ALGORITHM FOR CERTIFYING $\sum_k F(n,k) = f_n$

In Chapter 1, we found an upper bound for the order of the k-free linear recurrences with polynomial-in-n coefficients that the proper-hypergeometric terms satisfy. Now, we will apply the upper bound for J to Theorem 3.2A in [WZ3] to obtain an algorithm for finding directly the certificates, $a_0(n), a_1(n), \ldots, a_J(n)$, not all zero, and a rational function $R(n,k)$.

First, we state

Theorem 2.1. [WZ3, Theorem 3.2A] *Let F be a proper-hypergeometric term, and let $(n,k) \in \mathbb{Z}^2$ be a point at which $F(n,k) \neq 0$ and such that $F(n-j, k-i)$ is well-defined for all $0 \leq i \leq I$ and $0 \leq j \leq J$. Then there are polynomials $a_0(n), a_1(n), \ldots, a_J(n)$, not all zero, and a function $G(n,k)$ such that $G(n,k) = R(n,k)F(n,k)$ for some rational function R and such that*

$$(2.1)\quad a_0(n)F(n,k) + a_1(n)F(n-1,k) + \cdots + a_J(n)F(n-J,k) = G(n,k) - G(n,k-1).$$

The main idea of the algorithm is to find an upper bound $\mathcal{N}$ for the degree in k of the numerator polynomial of $R(n,k)$ from J^* in Theorem 1.4, for $R(n,k)$ must have the form

$$\frac{\sum_{i=0}^{\mathcal{N}} c_i(n)k^i}{D_R(n,k)},$$

where c_i's are polynomials in n. Knowing that we need at most $J^* + 1$ polynomials $a_j(n)$ for the recurrence and $\mathcal{N} + 1$ polynomials $c_i(n)$ for $R(n,k)$, we can solve for the a_j's and c_i's from a homogeneous linear system constructed in the algorithm.

ALGORITHM FOR THE CERTIFICATE

Step 1. Divide (2.1) by $F(n,k)$ to get

$$\sum_{j=0}^{J} a_j(n)\frac{F(n-j,k)}{F(n,k)} = R(n,k) - R(n,k-1)\frac{F(n,k-1)}{F(n,k)}. \tag{2.2}$$

Step 2. Find a common denominator for $R(n,k)$. From the proof of Theorem 3.2A in [WZ3], we know that $R(n,k)$ has the form

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J}\frac{\beta_{i,j}(n)F(n-j,k-i)}{F(n,k)}.$$

Therefore, a common denominator for $R(n,k)$ is $D_R(n,k) =$

$$P(n,k)\prod_{s=1}^{p}(a_s n + b_s k + c_s)^{\underline{(a_s)^+J+(b_s)^+(I-1)}}\prod_{s=1}^{q}(u_s n + v_s k + w_s + 1)^{\overline{(-u_s)^+J+(-v_s)^+(I-1)}}.$$

Step 3. Estimate the degree in k of the numerator polynomial $N_R(n,k)$ over the denominator $D_R(n,k)$. After some computation,

$$\deg_k N_R(n,k) = \deg_k D_R(n,k) + \max_{\substack{0\le i\le I-1\\ 0\le j\le J}}\Big(j\sum_{\substack{s\\ v_s\neq 0}} u_s + i\sum_s v_s - j\sum_{\substack{s\\ b_s\neq 0}} a_s - i\sum_s b_s\Big).$$

Let

$$U := \sum_{\substack{s\\ v_s\neq 0}} u_s,\quad V := \sum_s v_s,\quad A := \sum_{\substack{s\\ b_s\neq 0}} a_s,\quad B := \sum_s b_s,$$

$$\mathcal{A} := \Big(\sum_{\substack{s\\ b_s\neq 0}}(a_s)^+ + \sum_{\substack{s\\ v_s\neq 0}}(-u_s)^+\Big),\quad \mathcal{B} := \Big(\sum_s (b_s)^+ + \sum_s(-v_s)^+\Big).$$

We express $\deg_k N_R(n,k)$ in terms of the new variable names:

$$\deg_k N_R(n,k) = \deg_k P(n,k) + J\mathcal{A} + (I-1)\mathcal{B} + J(U-A)^+ + (I-1)(V-B)^+ =: \mathcal{N}.$$

Step 4. Assume that $R(n,k)$ has the form

$$\frac{\sum_{i=0}^{\mathcal{N}} c_i(n)k^i}{D_R(n,k)}.$$

Substitute it into (2.2) to get

$$\sum_{j=0}^{J} \frac{a_j(n)F(n-j,k)}{F(n,k)} - \sum_{i=0}^{\mathcal{N}} \frac{c_i(n)k^i}{D_R(n,k)} + \sum_{i=0}^{\mathcal{N}} \frac{c_i(n)(k-1)^i}{D_R(n,k-1)} \times \frac{F(n,k-1)}{F(n,k)} = 0. \tag{2.3}$$

Finally, the stage is set for solving for the unknown polynomials $a_j(n)$ and $c_i(n)$ for $0 \le j \le J$ and $0 \le i \le \mathcal{N}$.

Step 5. Find a common denominator for all three terms on the left of (2.3): A common denominator for (2.3) is

$$P(n,k) \prod_{s=1}^{p} (a_s n + b_s k + c_s)^{\underline{(a_s)^+ J + (b_s)^+ I}} \prod_{s=1}^{q} (u_s n + v_s k + w_s + 1)^{\overline{(-u_s)^+ J + (-v_s)^+ I}}$$
$$\times \prod_{s=1}^{p} (a_s n + b_s k + c_s)^{\underline{(a_s)^+}} \prod_{s=1}^{q} (u_s n + v_s k + w_s + 1)^{\overline{(-u_s)^+}}.$$

(From now on, we apply the same idea as in the proof of Theorem 3.1 in [WZ3].)

Step 6. With this common denominator, we find a common numerator of (2.3) and make the coefficient of every power of k that occurs in the common numerator polynomial vanish because (2.3) vanishes identically.

Step 7. Take the resulting system of linear homogeneous equations, and solve for the a_j's and c_i's. We know that a non-trivial solution exists from Theorem 3.1 of [WZ3].

CHAPTER III

THE r-VARIABLE CASE

In this chapter, we generalize the result of Chapter 1 to r summation indices. Definitions 3.1 and 3.2 are r-variable analogues of Definitions 1.1 and 1.2.

Notation. Let $\mathbf{k}$ be a vector in $\mathbb{Z}^r$. We use $\mathbf{z}^{\mathbf{k}}$ to denote $z_1^{k_1} z_2^{k_2} \cdots z_r^{k_r}$. For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^r$, $\mathbf{x} \cdot \mathbf{y}$ denotes the usual inner product. Define $\mathbf{x} \leq \mathbf{y}$ to mean $x_i \leq y_i$ for all $i \in [r]$. We use N_0 to denote the set $\{0, 1, \ldots\}$. As in Chapter 1, we let $x^{\underline{m}}$ denote $x(x-1)\cdots(x-m+1)$, and $x^{\overline{m}}$ denote $x(x+1)\cdots(x+m-1)$ for positive integers m. We define $x^{\underline{0}} = 1 = x^{\overline{0}}$.

Definition 3.1. A *proper-hypergeometric term* is a function of the form

$$F(n, \mathbf{k}) = P(n, \mathbf{k}) \frac{\prod_{s=1}^{p} (a_s n + \mathbf{b}_s \cdot \mathbf{k} + c_s)!}{\prod_{s=1}^{q} (u_s n + \mathbf{v}_s \cdot \mathbf{k} + w_s)!} \mathbf{z}^{\mathbf{k}}, \tag{3.1}$$

where P is a polynomial and $\mathbf{z}$ is a parameter. The a's, $\mathbf{b}$'s, u's and $\mathbf{v}$'s are assumed to be specific integers, i.e., they are integers and do not depend on any other parameters. The c's and the w's are also integers, but they may depend on parameters. We will say that F is *well-defined* at $(n, \mathbf{k})$ if none of the numbers $\{a_s n + \mathbf{b}_s \cdot \mathbf{k} + c_s\}_1^p$ is a negative integer. We will say that $F(n, \mathbf{k}) = 0$ if F is well-defined at $(n, \mathbf{k})$ and at least one of the numbers $\{u_s n + \mathbf{v}_s \cdot \mathbf{k} + w_s\}_1^q$ is a negative integer, or $P(n, \mathbf{k}) = 0$.

Definition 3.2. A proper-hypergeometric term F is said to satisfy a $\mathbf{k}$*-free recurrence* at a point $(n_0, \mathbf{k}_0) \in \mathbb{Z}^{r+1}$ if there are integers $I_1, I_2, \ldots, I_r$, J and polynomials $\alpha(\mathbf{i}, j, n)$ that do not depend on $\mathbf{k}$ and are not all zero, such that the relation

$$\sum_{\mathbf{i}=0}^{\mathbf{I}} \sum_{j=0}^{J} \alpha(\mathbf{i}, j, n) F(n-j, \mathbf{k}-\mathbf{i}) = 0 \tag{3.2}$$

holds for all $(n,\mathbf{k})$ in some $\mathbb{R}^{r+1}$ neighborhood of $(n_0,\mathbf{k}_0)$, in the sense that F is well-defined at all of the arguments that occur, and the relation (3.2) is true.

Theorem 3.3 [WZ3, Theorem 4.1]. *Every proper-hypergeometric term F in r variables satisfies a non-trivial* $\mathbf{k}$*-free recurrence relation. Indeed there exist I,J and polynomials $\alpha(\mathbf{i},j,n)$ $(\mathbf{i}=\mathbf{0},\ldots,\mathbf{I}; j=0,\ldots,J)$ not all zero, such that (3.2) holds at every point $(n_0,\mathbf{k}_0)\in\mathbb{Z}^{r+1}$ for which $F(n_0,\mathbf{k}_0)\neq 0$ and all of the values $F(n_0-j,\mathbf{k}_0-\mathbf{i})$ that occur in (3.2) are well-defined. Furthermore there exists such a recurrence in which $J=J^*$, where*

$$J^*=\left\lfloor\frac{1}{r!}\Big(\sum_{s=1}^{p}\sum_{r'=1}^{r}|(\mathbf{b}_s)_{r'}|+\sum_{s=1}^{q}\sum_{r'=1}^{r}|(\mathbf{v}_s)_{r'}|\Big)^r\right\rfloor.$$

Using the terminology and variable names of Theorem 3.3, we state

Theorem 3.4. *Let δ be the degree in* $\mathbf{k}$ *of $P(n,\mathbf{k})$, $2\leq r\in\mathrm{N}$, $\beta_i:=\mathcal{B}_i+(V_i-B_i)^+$, for $i\in[r]$, and $\beta_{r+1}:=\mathcal{A}+(U-A)^+$, where*

$$U:=\sum_{\substack{s\\ \mathbf{v}_s\neq\mathbf{0}}}u_s,\quad V_l:=\sum_s v_{ls},\quad A:=\sum_{\substack{s\\ \mathbf{b}_s\neq\mathbf{0}}}a_s,\quad B_l:=\sum_s b_{ls},$$

and

$$\mathcal{A}:=\sum_{\substack{s\\ \mathbf{b}_s\neq\mathbf{0}}}(a_s)^++\sum_{\substack{s\\ \mathbf{v}_s\neq\mathbf{0}}}(-u_s)^+,\quad \mathcal{B}_l:=\sum_s(b_{ls})^++\sum_s(-v_{ls})^+.$$

Furthermore, let

$$g(y):=y^{r+1}-\left(\prod_1^{r+1}\beta_i\right)\left(1+\left(\frac{\delta+r+(r+1)y-\sum_{i=1}^{r+1}\beta_i}{r}\right)\right).$$

The polynomial $g(y)$ has a zero that is greater than $2\max_i\{\beta_i\}$. If ρ_g denotes the largest zero of $g(y)$, then J^* in Theorem 3.3 can be replaced by

$$J^*=\left\lceil\frac{\rho_g}{\beta_{r+1}}\right\rceil-1.$$

3.1 Lemmas

We need the following lemmas for the proof of Theorem 3.4. The first lemma states that if $\mathbf{2} \le \mathbf{x} \in \mathbb{R}^{r+1}$ satisfies

$$\prod_{i=1}^{r+1} x_i \ge 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r},$$

then $\mathbf{x}$ is not at the boundary of the set $\{\mathbf{x} \mid \mathbf{x} \ge \mathbf{2}\}$. The second lemma states the existence of $\mathbf{x}^*$ subject to the inequality above such that $\boldsymbol{\beta} \cdot \mathbf{x}^*$ is a minimum of $\boldsymbol{\beta} \cdot \mathbf{x}$ and at the minimum,

$$\prod_{i=1}^{r+1} x_i = 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r}.$$

Lemma 3.5. *Let δ be a non-negative integer, $r \ge 2$ be a positive integer, and $\mathbf{1} \le \boldsymbol{\beta} \in \mathbb{R}^{r+1}$. If $\mathbf{2} \le \mathbf{x} \in \mathbb{R}^{r+1}$ satisfies*

$$\prod_{i=1}^{r+1} x_i \ge 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r},$$

then $\mathbf{x} > \mathbf{2}$.

Proof. Suppose not, say $x_{r+1} = 2$, then

$$2\prod_{i=1}^{r} x_i \ge 1 + \binom{r + \delta + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r}.$$

$$\begin{aligned}
\text{RHS} &= 1 + \prod_{j=1}^{r} \frac{j + \delta + \beta_{r+1} + \sum_1^r \beta_i (x_i - 1)}{j} \\
&= 1 + \prod_{j=1}^{r} \left(\frac{\sum_1^r \beta_i (x_i - 1)}{j} + 1 + \frac{\delta + \beta_{r+1}}{j} \right) \\
&= 1 + 2\left(\frac{\sum_1^r \beta_i (x_i - 1)}{2} + \frac{1 + \delta + \beta_{r+1}}{2} \right) \prod_{j=2}^{r} \left(\frac{\sum_1^r \beta_i (x_i - 1)}{j} + 1 + \frac{\delta + \beta_{r+1}}{j} \right).
\end{aligned}$$

Since $\beta_{r+1} \geq 1$, $\frac{1+\delta+\beta_{r+1}}{2} \geq 1$. Furthermore, $\frac{\sum_1^r \beta_i(x_i-1)}{j} \geq \frac{\sum_1^r \beta_i(x_i-1)}{r} \geq \frac{\sum_1^r (x_i-1)}{r}$ for all $2 \leq j \leq r$.

Since $(\frac{\sum_1^r x_i}{r})^r \geq \prod_{i=1}^r x_i$, RHS > LHS. A contradiction is reached upon assuming that one of the x_i's is 2. Therefore, we conclude that $\mathbf{x} > \mathbf{2}$. □

Lemma 3.6. *Let δ be a non-negative integer, and let $\boldsymbol{\beta} \in \mathbb{R}^{r+1}$ such that $\boldsymbol{\beta} \geq \mathbf{1}$. Then there exists $\mathbf{x}^* \geq \mathbf{2}$, $\mathbf{x}^* \in \mathbb{R}^{r+1}$ such that*

$$\boldsymbol{\beta} \cdot \mathbf{x}^* = \min \left\{ \boldsymbol{\beta} \cdot \mathbf{x} \,\middle|\, \mathbf{x} \in \mathbb{R}^{r+1}, \mathbf{x} \geq \mathbf{2}, \prod_{i=1}^{r+1} x_i \geq 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} \right\},$$

and

$$\prod_{i=1}^{r+1} x_i^* = 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x}^* - \mathbf{1})}{r}.$$

Proof. We first show the existence of $\mathbf{x}^*$. Choose a $\mathbf{y} \geq \mathbf{2}$ such that

$$\prod_{i=1}^{r+1} y_i \geq 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{y} - \mathbf{1})}{r}.$$

This is possible for a sufficiently large $\mathbf{y}$ because $\prod_1^{r+1} y_i$ is of degree $r+1$ and $\binom{\delta+r+\boldsymbol{\beta}\cdot(\mathbf{y}-\mathbf{1})}{r}$, of degree r. By Lemma 3.5, $\mathbf{y} > \mathbf{2}$. Consider the compact set

$$S = \{\mathbf{x} \,|\, 2 \leq x_i \leq \boldsymbol{\beta} \cdot \mathbf{y}, i \in [r+1]\}.$$

Note that $\mathbf{y} \in S$. We claim that if $\boldsymbol{\beta} \cdot \mathbf{x} \leq \boldsymbol{\beta} \cdot \mathbf{y}$, then $\mathbf{x} \in S$. Suppose $\mathbf{x} \notin S$, we show that $\boldsymbol{\beta} \cdot \mathbf{x} > \boldsymbol{\beta} \cdot \mathbf{y}$. If $\mathbf{x} \notin S$, then $x_i > \boldsymbol{\beta} \cdot \mathbf{y}$ for some $i \in [r+1]$. Since $\boldsymbol{\beta} \geq \mathbf{1}$, $\boldsymbol{\beta} \cdot \mathbf{x} > \beta_i(\boldsymbol{\beta} \cdot \mathbf{y}) + \sum_{j \neq i} \beta_j x_j > \boldsymbol{\beta} \cdot \mathbf{y}$.

Next we consider the closed set

$$T = \left\{ \mathbf{x} \,\middle|\, \mathbf{x} \geq \mathbf{2}, \prod_{i=1}^{r+1} x_i \geq 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} \right\}.$$

Clearly $S \cap T$ is compact and non-empty, for $\mathbf{y} \in S \cap T$. Furthermore,

$$\begin{aligned}\mathbf{x}^* &= \min_{\mathbf{x} \in S \cap T} \boldsymbol{\beta} \cdot \mathbf{x} \\ &= \min \left\{ \boldsymbol{\beta} \cdot \mathbf{x} \,\middle|\, \mathbf{x} \in \mathbb{R}^{r+1}, \mathbf{x} \geq \mathbf{2}, \prod_1^{r+1} x_i \geq 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} \right\},\end{aligned}$$

for $\prod_1^{r+1} x_i = 1 + \binom{\delta+r+\boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{1})}{r}$ is continuous in all x_i's.

Now we show that such an $\mathbf{x}^*$ satisfies

$$\prod_{i=1}^{r+1} x_i^* = 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x}^* - \mathbf{1})}{r}.$$

Suppose not, i.e., $\mathbf{x}^* = \min_{\mathbf{x} \in S \cap T} \boldsymbol{\beta} \cdot \mathbf{x}$, and $\prod_{i=1}^{r+1} x_i^* > 1 + \binom{\delta+r+\boldsymbol{\beta}\cdot(\mathbf{x}^*-\mathbf{1})}{r}$. Since $\mathbf{x}^* > \mathbf{2}$ by Lemma 3.5, there exists an open ball, hence a closed ball B centered at $\mathbf{x}^*$ in $\mathbb{R}^{r+1}$ such that

$$B = \left\{ \mathbf{x} \,\middle|\, \mathbf{x} > \mathbf{2}, \prod_1^{r+1} x_i > 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} \right\}.$$

But the map $\mathbf{x} \to \boldsymbol{\beta} \cdot \mathbf{x}$ is continuous, and B is compact. Therefore $\min_{\mathbf{x} \in B} \boldsymbol{\beta} \cdot \mathbf{x}$ is attained at the boundary of B. This leads to a contradiction, for $\mathbf{x}^*$ is not at the boundary of B. Thus at the minimum,

$$\prod_{i=1}^{r+1} x_i^* = 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x}^* - \mathbf{1})}{r}. \quad \square$$

3.2 A minimization problem

In the previous section, we proved the existence of $\mathbf{x}^* \geq \mathbf{2}$, $\mathbf{x}^* \in \mathbb{R}^{r+1}$ such that

$$\boldsymbol{\beta} \cdot \mathbf{x}^* = \min \left\{ \boldsymbol{\beta} \cdot \mathbf{x} \,\middle|\, \mathbf{x} \in \mathbb{R}^{r+1}, \mathbf{x} \geq \mathbf{2}, \prod_{i=1}^{r+1} x_i \geq 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} \right\},$$

and

$$\prod_{i=1}^{r+1} x_i^* = 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x}^* - \mathbf{1})}{r}.$$

In this section, we will express explicitly the minimum of $\boldsymbol{\beta} \cdot \mathbf{x}$ subject to the constraints $\mathbf{x} \geq \mathbf{2}$, and

$$\prod_{1}^{r+1} x_i \geq 1 + \binom{\delta + r + \sum \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r}$$

in terms of a root of a certain polynomial equation. (See (3.5) below.)

Proposition 3.7. *Let r be a natural number, let u and v be non-negative real numbers, and let w be a positive real number such that*

$$\frac{v}{r+1} \leq u \leq w.$$

For $\delta \geq 0$, define

$$f(x) := (x + 2u)^{r+1} - w\left(1 + \binom{\delta + r + (r+1)(x+2u) - v}{r}\right).$$

Then f has at most two positive zeros; if $2 \leq r \leq 4$, then f has one or two positive zeros; if $r \geq 5$, then f has exactly one positive zero.

Proof. We rewrite the expression for f:

$$f(x) = \sum_{k=0}^{r+1} \binom{r+1}{k} (2u)^k x^{r+1-k} - w - \frac{w}{r!} \prod_{i=1}^{r} \Big((r+1)x + 2u(r+1) - v + \delta + i\Big).$$

The case "$r = 1$" is utterly trivial: our function is a quadratic polynomial and, hence, has at most two (positive) roots. For $r \geq 2$ we prove the proposition using Descartes' rule of signs[1] [PS, p. 41]. Note that $[x^{r+1}]f(x) = \binom{r+1}{0}(2u)^0 = 1 > 0$.

[1] Descartes' rule of signs: A polynomial, $a_n x^n + \cdots + a_1 x + a_0$, has at most as many positive zeros as there are sign changes in the sequence $a_0, a_1, \ldots, a_n$, or less by an even number.

If $r = 2$,

$$\begin{aligned} f(x) &= x^3 + 6ux^2 + 12u^2x + 8u^3 - w \\ &\quad - \tfrac{w}{2}(3x + 6u - v + \delta + 1)(3x + 6u - v + \delta + 2) \\ &= x^3 + 6ux^2 - \tfrac{9}{2}wx^2 \\ &\quad + 12u^2x - \tfrac{w}{2}3x(12u - 2v + 2\delta + 3) \\ &\quad + 8u^3 - w - \tfrac{w}{2}(6u - v + \delta + 1)(6u - v + \delta + 2). \end{aligned}$$

We see that

$$[x^2]f(x) = 6u - \tfrac{9}{2}w \quad \begin{cases} > 0 & \text{if } u > \tfrac{3}{4}w; \\ < 0 & \text{if } u < \tfrac{3}{4}w. \end{cases}$$

If $u > \frac{3}{4}w$, then the two leading coefficients are positive, and it follows from Descartes' rule that f has at most two positive zeros. If $u \le \frac{3}{4}w$, then

$$\begin{aligned} [x]f(x) &= 12u^2 - \tfrac{w}{2}3(12u - 2v + 2\delta + 3) \\ &\le 12u^2 - 2u(12u - 2v + 2\delta + 3) \\ &< 12u^2 - 2u(12u - 2v) \\ &\le 12u^2 - 2u6u = 0 \end{aligned}$$

because $v \le (r + 1)u = 3u$. The list of coefficients therefore reads: positive, negative, negative, unknown. It again follows from Descartes' rule that f has at most two positive zeros.

Now to the general case, "$r \ge 3$." By assumption, $u(r + 1) \ge v$, so $2u(r + 1) - v + \delta \ge$

$u(r+1)+\delta \geq u(r+1) \geq 0$. For $1 \leq k \leq r+1$ we therefore get that

$$
\begin{aligned}
(3.3) \qquad & [x^{r+1-k}] \prod_{i=1}^{r} \Big((r+1)x + \big(2u(r+1) - v + \delta + i\big) \Big) \\
&= (r+1)^{r+1-k} \sum_{\substack{S \subseteq [r] \\ |S| = k-1}} \prod_{i \in S} \big(2u(r+1) - v + \delta + i\big) \\
&> (r+1)^{r+1-k} \binom{r}{k-1} \big(2u(r+1) - v + \delta\big)^{k-1} \\
&\geq (r+1)^{r+1-k} \binom{r}{k-1} \big(u(r+1)\big)^{k-1} \\
&= \binom{r}{k-1} (r+1)^r u^{k-1}.
\end{aligned}
$$

It follows, for $1 \leq k \leq r$, that

$$
\begin{aligned}
[x^{r+1-k}]f(x) &< \binom{r+1}{k} (2u)^k - \frac{w}{r!} \binom{r}{k-1} (r+1)^r u^{k-1} \\
&= \frac{(r+1)!}{k!\,(r+1-k)!} 2^k u^k - \frac{w}{r!} \frac{r!}{(k-1)!\,(r+1-k)!} (r+1)^r u^{k-1} \\
&= \frac{(r+1)u^{k-1}}{(k-1)!\,(r+1-k)!} \left(\frac{2^k r!}{k} u - w(r+1)^{r-1} \right) \\
&< \frac{(r+1)u^{k-1}}{(k-1)!\,(r+1-k)!} \left(\frac{2^r r!}{r} u - w(r+1)^{r-1} \right) \\
&\leq \frac{(r+1)u^{k-1}}{(k-1)!\,(r+1-k)!} \left(2^r (r-1)!\, u - u(r+1)^{r-1} \right) < 0
\end{aligned}
$$

because $u \leq w$ and $2^r \leq \frac{(r+1)^{r-1}}{(r-1)!}$ for $r \geq 3$. (To prove this, note that $\frac{(r+1)^r}{r!} / \frac{r^{r-1}}{(r-1)!} = (\frac{r+1}{r})^r = (1+\frac{1}{r})^r = \sum_{i=0}^{r} \binom{r}{i} (\frac{1}{r})^r = 1 + r\frac{1}{r} + \cdots \geq 2$. Since $2^{5+1} < \frac{(5+1)^5}{5!}$, it follows that

$$
(3.4) \qquad 2^{r+1} \leq \frac{(r+1)^r}{r!} \qquad \text{for } r \geq 5.
$$

Thus $2^r \leq \frac{r^{r-1}}{(r-1)!} < \frac{(r+1)^{r-1}}{(r-1)!}$ for $r \geq 6$; the cases $r = 3, 4, 5$ are easily checked.)

We have shown that $[x^{r+1-k}]f(x) < 0$ for $1 \leq k \leq r$, whence f has at most two positive zeros by Descartes' rule.

Now assume that $r \geq 5$. Using first (3.3) and then (3.4) we get that

$$\begin{aligned}[x^0]f(x) &< \binom{r+1}{r+1}(2u)^{r+1} - w - \frac{w}{r!}\binom{r}{r}(r+1)^r u^r \\ &= 2^{r+1}u^{r+1} - w - wu^r\frac{(r+1)^r}{r!} \\ &\leq 2^{r+1}u^{r+1} - w - wu^r 2^{r+1} < 0\end{aligned}$$

because $u \leq w$. It follows from Descartes' rule that f has at most one positive zero, and clearly—the constant term being negative and the leading coefficient being positive—there is at least one positive zero. □

Corollary 3.8. *For integers $\delta \geq 0$, $r \geq 2$, and $1 \leq \beta_{r+1} \leq \beta_r \leq \cdots \leq \beta_1$, define*

$$f(x) := (x + 2\beta_1)^{r+1} - \left(1 + \binom{\delta + r + (r+1)(x+2\beta_1) - \sum_{i=1}^{r+1}\beta_i}{r}\right)\prod_{i=1}^{r+1}\beta_i.$$

If $r = 3$ or 4, then f has at most two positive zeros; otherwise, f has exactly one positive zero.

Proof. Let $u := \beta_1$, $v := \sum_{i=1}^{r+1}\beta_i$, and $w := \prod_{i=1}^{r+1}\beta_i$. Clearly u, v, and w are positive real numbers and $\frac{v}{r+1} \leq u \leq w$. By Proposition 3.7, f has at most two positive zeros, and f has exactly one positive zero, if $r \geq 5$. We need only deal with the case '$r = 2$'. To do so, we expand $f(x)$ as in the proof of Proposition 3.7 to read off the coefficients. Note that $[x^3]f(x) = 1 > 0$.

If $u \neq w$, then $\frac{w}{u} \geq 2$ because all the β_i's are positive integers. Hence

$$\begin{aligned}[x^2]f(x) &= 6u - \tfrac{9}{2}w < 0; \\ [x]f(x) &= 12u^2 - \tfrac{3}{2}w(12u - 2v + 2\delta + 3) \\ &< 12u^2 - \tfrac{3}{2}w(6u + 2\delta + 3) < 0; \\ [x^0]f(f) &= 8u^3 - w - \tfrac{w}{2}(6u - v + \delta + 1)(6u - v + \delta + 2) \\ &\leq 8u^3 - w - \tfrac{w}{2}9u^2 < 0.\end{aligned}$$

If $u = w$, then $\beta_i = 1$ for all $i \in [r+1]$, so $u = v = w$. Thus

$$f(x) = x^3 + \tfrac{3}{2}x^2 - \left(\tfrac{15}{2} + 3\delta\right)x - \left(14 + \tfrac{13}{2}\delta + \tfrac{\delta^2}{2}\right).$$

In either case, there is only one sign change in the sequence of coefficients of $f(x)$, so f has at most one positive zero by Descartes' rule; and clearly—the leading coefficient being positive and the constant term being negative (in both cases)—our function has at least one positive zero. □

Example 3.9. Let $\delta = 0$, $r = 3$, $u = 4$, $v = 7$ and $w = 4$, then $f(x)$ has two positive zeros.

Theorem 3.10. *Let δ be a non-negative integer, and $2 \le r \in \mathbb{Z}$. Let $\boldsymbol{\beta} \ge \mathbf{1}$, $\boldsymbol{\beta} \in \mathbb{R}^{r+1}$. The minimum of $\boldsymbol{\beta} \cdot \mathbf{x}$*

$$\text{subject to} \quad \mathbf{x} \ge \mathbf{2}$$
$$\text{and} \quad \prod_{1}^{r+1} x_i \ge 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r}$$

is $(r+1)y^*$, where y^* is the smallest zero of

$$g(y) = \frac{y^{r+1}}{\prod_{i=1}^{r+1} \beta_i} - 1 - \binom{\delta + r + (r+1)y - \sum_{i=1}^{r+1} \beta_i}{r} \tag{3.5}$$

such that $y^* > 2\max_i\{\beta_i\}$.

Proof. For all $c \ge \sum_i \beta_i$, we define $H(c)$ to be the hyperplane

$$\{\mathbf{x} \mid \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1}) = c\}.$$

Also we define a closed set

$$T = \left\{ \mathbf{x} \,\middle|\, \mathbf{x} \ge \mathbf{2}, \prod_{i=1}^{r+1} x_i \ge 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} \right\}.$$

By Lemma 3.5, the boundary of T is

$$\partial T = \left\{ \mathbf{x} \,\middle|\, \mathbf{x} \geq \mathbf{2}, \prod_{i=1}^{r+1} x_i = 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} \right\}.$$

To minimize $\boldsymbol{\beta} \cdot \mathbf{x}$ over $\mathbf{x} \in T$ is equivalent to finding the smallest value c such that $H(c) \cap T$ is not empty. Since Lemma 3.6 asserts that the minimum c occurs at the boundary of T, we are looking for the smallest c such that $H(c) \cap \partial T \neq \varnothing$. In other words, our problem is to

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) := \boldsymbol{\beta} \cdot \mathbf{x} \\ \text{subject to} & \mathbf{x} \geq \mathbf{2} \\ \text{and} & g(\mathbf{x}) := \prod_1^{r+1} x_i - 1 - \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} = 0. \end{array}$$

Since f and g are both continuous functions in $\mathbf{x}$, and Lemma 3.5 tells us that $\mathbf{x} > \mathbf{2}$, the conditions for using Lagrange multiplier rule [MP, p. 360–363] are satisfied. We find $\lambda \neq 0$ and $\mathbf{x}_0 > \mathbf{2}$ such that $\lambda \nabla f(\mathbf{x}_0) = \nabla g(\mathbf{x}_0)$.

Let us first compute ∇f and ∇g. Since $f(\mathbf{x}) = \boldsymbol{\beta} \cdot \mathbf{x}$, $\nabla f(\mathbf{x}) = \boldsymbol{\beta}$. Since

$$g(\mathbf{x}) := \prod_1^{r+1} x_i - 1 - \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r} = 0,$$

we have $\nabla g(\mathbf{x}) = \boldsymbol{\gamma}$, where

$$\gamma_j = \frac{\prod_1^{r+1} x_i}{x_j} - \frac{\beta_j}{r!} \prod_{i=1}^{r} (\delta + i + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})) \left(\sum_{i=1}^{r} \frac{1}{i + \delta + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})} \right).$$

To solve for λ and $\mathbf{x}_0$ in the equation, $\lambda \nabla f(\mathbf{x}_0) = \nabla g(\mathbf{x}_0)$, we set $\lambda \beta_j = \gamma_j$ for all $j \in [r+1]$, or equivalently,

$$\beta_j x_j = \frac{\prod_1^{r+1} x_i}{\lambda + \frac{\prod_i (\delta + i + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1}))}{r!} \left(\sum_{i=1}^{r} \frac{1}{i + \delta + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})} \right)},$$

for all $j \in [r+1]$. Note that the right hand side of the last equality is the same for all $j \in [r+1]$. Thus at the minimum, $\beta_j x_j = \beta_i x_i$ for all $i, j \in [r+1]$, and

$$\lambda = \frac{\prod_1^{r+1} x_i}{\beta_j x_j} - \frac{\prod_{i=1}^{r}(\delta + i + \boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{1}))}{r!}\left(\sum_{i=1}^{r}\frac{1}{i+\delta+\boldsymbol{\beta}\cdot(\mathbf{x}-\mathbf{1})}\right).$$

Let $y := \beta_j x_j$ for $j \in [r+1]$. We substitute y into $g(\mathbf{x})$ to get

$$g(y) = \frac{y^{r+1}}{\prod_{i=1}^{r+1}\beta_i} - 1 - \binom{\delta + r + (r+1)y - \sum_{i=1}^{r+1}\beta_i}{r}. \tag{3.5}$$

If y^* is the smallest zero of $g(y)$ such that $y^* > 2\max_i\{\beta_i\}$, then the minimum of f is $(r+1)y^*$. By Corollary 3.8, we know that such a y^* exists. In the case where $r \neq 3$ or 4, $g(y)$ has only one zero $> 2\max_i\{\beta_i\}$. (Since $u = \max_i\{\beta_i\}$ and $w = \prod_i \beta_i$, u divides w. Thus it can be shown that when $r = 2$, $f(x)$ has only one positive zero.) □

3.3 Proof of Theorem 3.4

In this section, we use Theorem 3.10 to estimate sharper upper bounds for the I_i's and J from Theorem 3.3.

Proof of Theorem 3.4. Let $\hat{F}(n,\mathbf{k}) := F(n,\mathbf{k})/P(n,\mathbf{k})$. Fix some $I_1, I_2, \ldots, I_r$, $J > 0$, and suppose $(n_0, \mathbf{k}_0)$ is a point that satisfies the two conditions of the theorem. Since we assumed that all of the $a_s, \mathbf{b}_s, u_s, \mathbf{v}_s$ in Definition 3.1 are integers, we have that for all $(n,\mathbf{k})$ in some $\mathbb{R}^{r+1}$ neighborhood of $(n_0,\mathbf{k}_0)$, all of the ratios $F(n-j,\mathbf{k}-\mathbf{i})/\hat{F}(n,\mathbf{k})$ are well-defined *rational functions* of n and $\mathbf{k}$. (See (3.1) for $F(n,\mathbf{k})$.) Hence we can form a linear combination

$$W(\mathbf{k}) := \sum_{\mathbf{i}=0}^{\mathbf{I}}\sum_{j=0}^{J}\alpha(\mathbf{i},j,n)\frac{F(n-j,\mathbf{k}-\mathbf{i})}{\hat{F}(n,\mathbf{k})} \tag{3.6}$$

of these rational functions, in which the α's are to be determined, if possible, so as to make the sum vanish identically in the neighborhood.

As in [WZ3], the problem is to find a common denominator for the summand in (3.6). Instead, we find a common denominator $D(n,\mathbf{k})$ for $\sum_{\mathbf{i}=0}^{\mathbf{I}}\sum_{j=0}^{J}\frac{F(n-j,\mathbf{k}-\mathbf{i})}{\hat{F}(n,\mathbf{k})}$. Clearly, $D(n,\mathbf{k})$ is also a common denominator for the summand in (3.6). Consider

$$\frac{F(n-j,\mathbf{k}-\mathbf{i})}{\hat{F}(n,\mathbf{k})} = P(n-j,\mathbf{k}-\mathbf{i})\mathbf{z}^{-\mathbf{i}}\prod_{s=1}^{p}\frac{(a_s n+\mathbf{b}_s\cdot\mathbf{k}+c_s-a_s j-\mathbf{b}_s\cdot\mathbf{i})!}{(a_s n+\mathbf{b}_s\cdot\mathbf{k}+c_s)!} \times\prod_{s=1}^{q}\frac{(u_s n+\mathbf{v}_s\cdot\mathbf{k}+w_s)!}{(u_s n+\mathbf{v}_s\cdot\mathbf{k}+w_s-u_s j-\mathbf{v}_s\cdot\mathbf{i})!} \tag{3.7}$$

which contributes to the denominator $D(n,\mathbf{k})$, if

$$a_s j+\mathbf{b}_s\cdot\mathbf{i}>0,\quad\text{and/or}\quad u_s j+\mathbf{v}_s\cdot\mathbf{i}<0.$$

Let $A_s := a_s n+\mathbf{b}_s\cdot\mathbf{k}+c_s$ and $U_s := u_s n+\mathbf{v}_s\cdot\mathbf{k}+w_s$. In (3.7), if $a_s j+\mathbf{b}_s\cdot\mathbf{i}>0$ for some $s\in[p]$, then

$$\frac{A_s!}{(A_s-a_s j-\mathbf{b}_s\cdot\mathbf{i})!}\quad\text{divides}\quad\frac{A_s!}{(A_s-\max\limits_{\substack{0\le j\le J\\ 0\le\mathbf{i}\le\mathbf{I}}}(a_s j+\mathbf{b}_s\cdot\mathbf{i})^+)!}.$$

But

$$\max_{\substack{0\le j\le J\\ 0\le\mathbf{i}\le\mathbf{I}}}(a_s j+\mathbf{b}_s\cdot\mathbf{i})^+ = (a_s)^+J+\sum_{l=1}^{r}(b_{ls})^+I_l.$$

Similarly, if $u_s j+\mathbf{v}_s\cdot\mathbf{i}<0$ for some $s\in[q]$, then

$$\frac{(U_s-u_s j-\mathbf{v}_s\cdot\mathbf{i})!}{U_s!}\quad\text{divides}\quad\frac{(U_s+\max\limits_{\substack{0\le j\le J\\ 0\le\mathbf{i}\le\mathbf{I}}}(-u_s j-\mathbf{v}_s\cdot\mathbf{i})^+)!}{U_s!},$$

and, as before,

$$\max_{\substack{0\le j\le J\\ 0\le\mathbf{i}\le\mathbf{I}}}(-u_s j-\mathbf{v}_s\cdot\mathbf{i})^+ = (-u_s)^+J+\sum_{l=1}^{r}(-v_{ls})^+I_l.$$

A common denominator for $W(\mathbf{k})$ (see (3.6)) is

$$D(n,\mathbf{k}) = \prod_{s=1}^{p} (A_s)^{\overline{(a_s)^+ J + \sum_{l=1}^{r} (b_{ls})^+ I_l}} \prod_{s=1}^{q} (U_s + 1)^{\overline{(-u_s)^+ J + \sum_{l=1}^{r} (-v_{ls})^+ I_l}}.$$

Thus the degree in $\mathbf{k}$ of $D(n,\mathbf{k})$ is

$$(3.8) \quad J\Big(\sum_{\substack{s\in[p]\\ \mathbf{b}_s\neq 0}} (a_s)^+\Big) + \sum_{l=1}^{r}\Big(I_l \sum_{s\in[p]} (b_{ls})^+\Big) + J\Big(\sum_{\substack{s\in[q]\\ \mathbf{v}_s\neq 0}} (-u_s)^+\Big) + \sum_{l=1}^{r}\Big(I_l \sum_{s\in[q]} (-v_{ls})^+\Big)$$

$$= J\Big(\sum_{\substack{s\in[p]\\ \mathbf{b}_s\neq 0}} (a_s)^+ + \sum_{\substack{s\in[q]\\ \mathbf{v}_s\neq 0}} (-u_s)^+\Big) + \sum_{l=1}^{r} I_l\Big(\sum_{s\in[p]} (b_{ls})^+ + \sum_{s\in[q]} (-v_{ls})^+\Big).$$

Next, we find the degree in $\mathbf{k}$ of the common numerator polynomial $N(n,\mathbf{k})$ of $W(\mathbf{k})$ of (3.6) after using the common denominator $D(n,\mathbf{k})$ above. Consider the $(\mathbf{i},j)$th term in

$$\sum_{\mathbf{i}=0}^{\mathbf{I}} \sum_{j=0}^{J} \frac{F(n-j,\mathbf{k}-\mathbf{i})}{\hat{F}(n,\mathbf{k})}.$$

After the same computation as in Chapter 1, we get that the degree in $\mathbf{k}$ of the numerator polynomial of the $(\mathbf{i},j)$th term over $D(n,\mathbf{k})$ is

$$\sum_{\mathbf{b}_s\neq 0} (-a_s j - \mathbf{b}_s\cdot\mathbf{i})^+ + \sum_{\mathbf{v}_s\neq 0} (u_s j + \mathbf{v}_s\cdot\mathbf{i})^+$$
$$+ \deg_{\mathbf{k}} D(n,\mathbf{k}) - \sum_{\mathbf{b}_s\neq 0} (a_s j + \mathbf{b}_s\cdot\mathbf{i})^+ - \Big(\sum_{\mathbf{v}_s\neq 0} (-u_s j - \mathbf{v}_s\cdot\mathbf{i})^+\Big) + \delta$$
$$= \deg_{\mathbf{k}} D(n,\mathbf{k}) + \delta + \sum_{\mathbf{v}_s\neq 0} (u_s j + \mathbf{v}_s\cdot\mathbf{i}) - \sum_{\mathbf{b}_s\neq 0} (a_s j + \mathbf{b}_s\cdot\mathbf{i}),$$

where $\delta := \deg_{\mathbf{k}} P(n,\mathbf{k})$. Therefore, the degree in $\mathbf{k}$ of the common numerator polynomial in $W(\mathbf{k})$ is

$$\deg_{\mathbf{k}} N(n,\mathbf{k})$$
$$= \delta + \max_{\mathbf{i},j}\Big(\deg_{\mathbf{k}} D(n,\mathbf{k}) + \sum_{\mathbf{v}_s\neq 0} (u_s j + \mathbf{v}_s\cdot\mathbf{i}) - \sum_{\mathbf{b}_s\neq 0} (a_s j + \mathbf{b}_s\cdot\mathbf{i})\Big)$$
$$= \delta + \deg_{\mathbf{k}} D(n,\mathbf{k}) + \max_{\mathbf{i},j}\Big(j \sum_{\substack{s\\ \mathbf{v}_s\neq 0}} u_s + \mathbf{i}\cdot\sum_{s} \mathbf{v}_s - j \sum_{\substack{s\\ \mathbf{b}_s\neq 0}} a_s - \mathbf{i}\cdot\sum_{s} \mathbf{b}_s\Big).$$

Let

$$U := \sum_{\substack{s \\ \mathbf{v}_s \neq 0}} u_s, \quad V_l := \sum_s v_{ls}, \quad A := \sum_{\substack{s \\ \mathbf{b}_s \neq 0}} a_s, \quad B_l := \sum_s b_{ls}.$$

We can rewrite $\deg_{\mathbf{k}} N(n, \mathbf{k})$ as

$$\begin{aligned} \deg_{\mathbf{k}} N(n, \mathbf{k}) &= \delta + \deg_{\mathbf{k}} D(n, \mathbf{k}) + \max_{\mathbf{i}, j} \Big(j(U - A) + \sum_l i_l (V_l - B_l) \Big) \\ &= \delta + \deg_{\mathbf{k}} D(n, \mathbf{k}) + J(U - A)^+ + \sum_l I_l (V_l - B_l)^+. \end{aligned}$$

To simplify the expression for $\deg_{\mathbf{k}} N(n, \mathbf{k})$, we let

$$\mathcal{A} := \sum_{\substack{s \\ \mathbf{b}_s \neq 0}} (a_s)^+ + \sum_{\substack{s \\ \mathbf{v}_s \neq 0}} (-u_s)^+, \quad \mathcal{B}_l := \sum_s (b_{ls})^+ + \sum_s (-v_{ls})^+.$$

Then, substituting the expression for $\deg_{\mathbf{k}} D(n, \mathbf{k})$ in (3.8), we get

$$\begin{aligned} \deg_{\mathbf{k}} N(n, \mathbf{k}) &= \delta + J\mathcal{A} + \sum_l I_l \mathcal{B}_l + J(U - A)^+ + \sum_l I_l (V_l - B_l)^+ \\ &= \delta + J(\mathcal{A} + (U - A)^+) + \sum_l I_l (\mathcal{B}_l + (V_l - B_l)^+) \\ &= \delta + \sum_{j=1}^{r} \beta_j I_j + \beta_{r+1} J, \end{aligned}$$

where $\beta_j := \mathcal{B}_j + (V_j - B_j)^+$ for $j \in [r]$, and $\beta_{r+1} := \mathcal{A} + (U - A)^+$.

Knowing the degree in $\mathbf{k}$ of $N(n, \mathbf{k})$, we deduce that there are at most $\binom{\deg_{\mathbf{k}} N(n,\mathbf{k}) + r}{r}$ homogeneous linear equations to solve in $(I_1 + 1)(I_2 + 1) \cdots (I_r + 1)(J + 1)$ unknowns, namely, the $\alpha(\mathbf{i}, j, n)$'s. A system of solutions for $\alpha(\mathbf{i}, j, n)$'s exists, if

$$(I_1 + 1)(I_2 + 1) \cdots (I_r + 1)(J + 1) \geq 1 + \binom{\deg_{\mathbf{k}} N(n, \mathbf{k}) + r}{r}.$$

In order to obtain good upper bounds for $\mathbf{I}$ and J, we minimize $\deg_{\mathbf{k}} N(n, \mathbf{k})$ subject to the condition just stated.

With a change of variables, $x_i := I_i + 1$, $I_i \geq 1$, for all $i \in [r]$, and $x_{r+1} := J + 1$, we can state our task more easily as:

$$\begin{aligned} &\text{Minimize} && \boldsymbol{\beta} \cdot \mathbf{x} \\ &\text{subject to} && \mathbf{x} \geq \mathbf{2} \\ &\text{and} && \prod_{1}^{r+1} x_i \geq 1 + \binom{\delta + r + \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{1})}{r}. \end{aligned}$$

Let us suppose that one of $\{\beta_i\}_{i\in[r+1]}$ is zero. Say $\beta_l = 0$ for some $l \in [r+1]$. This means that

$$\begin{aligned} \beta_l &= \mathcal{B}_l + (V_l - B_l)^+ \\ &= \sum_s (b_{ls})^+ + \sum_s (-v_{ls})^+ + \left(\sum_s v_{ls} - \sum_s b_{ls}\right)^+ \\ &= 0. \end{aligned}$$

Hence $b_{ls} = 0$ for all $s \in [p]$, and $v_{ls} = 0$ for all $s \in [q]$. Therefore, k_l is absent in the factorial part of $F(n, \mathbf{k})$.

If the variable k_l actually appears in $P(n, \mathbf{k})$, then the summation of $F(n, \mathbf{k})$ is infinite. Since we consider only terminating hypergeometric identities, we will assume that the variable k_l is absent in $F(n, \mathbf{k})$ if it is absent in the products of factorials in $F(n, \mathbf{k})$. In other words, $F(n, \mathbf{k})$ is independent of k_l, if $\beta_l = 0$.

Henceforth, we consider only the case where β_i, $(i \in [r+1])$ are positive integers. Chapter 1 dealt with the case where $r = 1$. Thus we assume that $r \geq 2$. From Theorem 3.10, we conclude that $\boldsymbol{\beta} \cdot \mathbf{x}$ attains its minimum at

$$x_i^* = \frac{y^*}{\beta_i},$$

where y^* is the smallest zero greater than $2\max\{\beta_i\}$ of

$$g(y) = \frac{y^{r+1}}{\prod_{i=1}^{r+1}\beta_i} - 1 - \binom{\delta + r + (r+1)y - \sum_{i=1}^{r+1}\beta_i}{r}.$$

If all x_i^*'s thus obtained are integers, then we are done: the upper bounds for the I_i^* are $x_i^* - 1$. In particular $J^* = x_{r+1}^* - 1$. Otherwise, let ρ_g be the largest zero of g. (In the case when $r \neq 3$ or 4, ρ_g is the only zero of g greater than $2\max_i\{\beta_i\}$ —see Corollary 3.8.) Since $g(y) \to \infty$ as $y \to \infty$, $g(y) > 0$ for all $y > \rho_g$. A bound for x_i^* is

$$x_i^* \leq \left\lceil \frac{\rho_g}{\beta_i} \right\rceil.$$

Thus,

$$I_i^* = \left\lceil \frac{\rho_g}{\beta_i} \right\rceil - 1,$$

and

$$J^* = \left\lceil \frac{\rho_g}{\beta_{r+1}} \right\rceil - 1. \quad \square$$

Remarks. We first remark that the J^* thus obtained is not always better than the bound from Chapter 1. A simple calculation shows that in Example 1.6, J^* of [W2] is m, whereas we get $J^* = 2m-2$. However, Theorem 3.4 gives the best overall bounds, when all I's and J are considered. For example, in Example 1.6, $J^* = m$ and $I^* = (m-1)m+1$, whereas $J^* = 2m-2$ and $I^* = 2m-1$ using the method of Theorem 3.4.

The second remark is about the size of ρ_g. From Formula 14 of [W1], we know that

$$\rho_g \leq 2 \max_{i\in[r+1]} \left| \frac{d_{r+1-i}}{d_{r+1}} \right|^{\frac{1}{i}},$$

where $d_i = [y^i]g(y)$.

The last remark is about how to find a better bound for I_i^*. The following algorithm takes one or two zeros of g that are greater than $2\max_i\{\beta_i\}$ and tests the feasibility of x_i^* between the choices $\lfloor \frac{y}{\beta_i} \rfloor$ and $\lceil \frac{y}{\beta_i} \rceil$.

ALGORITHM 1. Algorithm for finding better upper bounds

Input: y_1 (if $r = 2$ or $r \geq 5$); y_1, y_2 (if $r = 3$ or 4)

```
j:=1
repeat
    for i := 1 to r+1
        x_i^* := y_i / β_i
    next i
    R := {i | x_i^* ∉ Z, i ∈ [r+1]}
    {x_k^*}_{k=1}^{2^|R|} := {x^* | x_i^* = ⌊y_i/β_i⌋ + ε_i; ε_i = 0 if i ∉ R, ε_i = 1 if i ∈ R}
    k := 0
    repeat
        k := k+1
    until ∏ x_i^* ≥ 1 + binom(δ+r+β·(x_i^*−1), r) or k = 2^|R| + 1
    j := j+1
until ∏ x_i^* ≥ 1 + binom(δ+r+β·(x^*−1), r) or j = 2+1
```

Output: $\mathbf{x}^*$

CHAPTER IV

AN ALGORITHM FOR CERTIFYING $\sum_{\mathbf{k}} F(n, \mathbf{k}) = f_n$

In this chapter, we will develop the r-variable analogues of the algorithm in Chapter 2. We will take the values of $I_1^*, I_2^*, \ldots, I_r^*$ and J^* from Chapter 3, and input them into the algorithm to obtain directly $a_0(n), a_1(n), \ldots, a_J(n)$, not all zero, and rational functions $R_1(n, \mathbf{k}), R_2(n, \mathbf{k}), \ldots, R_r(n, \mathbf{k})$.

Here Theorem 4.2A, the analogue of Theorem 3.2A in [WZ3] is what we need to construct the algorithm.

Theorem 4.1. [WZ3, Theorem 4.2A] *Let F be a proper-hypergeometric term. Then there are a positive integer J, polynomials $a_0(n), a_1(n), \ldots, a_J(n)$ and hypergeometric functions $G_1, \ldots, G_r$ such that for every $(n, \mathbf{k}) \in \mathbf{N}^{r+1}$ at which $F \neq 0$ and F is well-defined at all of the arguments that appear in*

$$\sum_{\mathbf{i}=0}^{\mathbf{I}} \sum_{j=0}^{J} \alpha(\mathbf{i}, j, n) F(n-j, \mathbf{k}-\mathbf{i}) = 0 \tag{4.1}$$

we have

$$\sum_{j=0}^{J} a_j(n) F(n-j, \mathbf{k}) = \sum_{i=1}^{r} \big(G_i(n, k_1, \ldots, k_i, \ldots, k_r) - G_i(n, k_1, \ldots, k_i - 1, \ldots, k_r) \big). \tag{4.2}$$

Moreover this recurrence is non-trivial, and each $G_i(n, \mathbf{k})$ is of the form $R_i(n, \mathbf{k}) F(n, \mathbf{k})$, where the R's are rational functions of their arguments.

In the proof of Theorem 4.1 in [WZ3], we let N be the operator that shifts (down) the variable n, that is $Nf(n) = f(n-1)$. Further, for each $i = 1, \ldots, r$ we let K_i be the operator that shifts the variable k_i, that is $K_i f(\mathbf{k}) = f(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$. Then (4.1) is equivalent to an assertion

$$H(N, n, K_1, \ldots, K_r)F(n, \mathbf{k}) = 0,$$

where H is a polynomial in its arguments and does not involve $\mathbf{k}$. We can expand H in a Taylor's series about $\mathbf{K} = \mathbf{1}$, to obtain

$$H(N, n, \mathbf{K}) = H(N, n, \mathbf{1}) + \sum_{i=1}^{r}(K_i - 1)V_i(N, n, \mathbf{K})$$

in which the V_i's are polynomials in their arguments. We apply the right hand side of the last equality to $F(n, \mathbf{k})$, and (4.2) follows.

We generalize the idea from Chapter 2 to the multivariable case. From Chapter 3, we can compute the upper bounds for $\mathbf{I}^*$ and J^* which are used to find the degree in $\mathbf{k}$ of the numerator polynomial of $R_i(n, \mathbf{k})$, $i \in [r]$. Into (4.2), we substitute

$$\sum_{\substack{0 \leq \mathbf{e} \leq (\mathcal{N}_i, \mathcal{N}_i, \ldots, \mathcal{N}_i) \\ \mathbf{e} \cdot \mathbf{1} \leq \mathcal{N}_i}} \frac{c_i(\mathbf{e}, n)\mathbf{k}^{\mathbf{e}}}{D_{R_i}(n, \mathbf{k})}$$

for $R_i(n, \mathbf{k})$ where the $c_i(\mathbf{e}, n)$ are unknown polynomials in n. The procedure that follows yields a homogeneous linear system with the c_i's and a_j's as the unknowns for which a solution is guaranteed by [WZ3, Theorem 4.1].

Algorithm for the Certificate

Step 1. Rename k_i's so that $I_1 \geq I_2 \geq \cdots \geq I_r$ for the given I_i's.

Step 2. Obtain $\sum_{i=1}^{r}(K_i - 1)V_i(N, n, \mathbf{K})$ in the following way. For the Taylor's series expansion of $H(N, n, \mathbf{K})$ about $\mathbf{K} = \mathbf{1}$, we first sum all terms that are divisible by (K_1-1), factor (K_1-1) to make the sum equal to $(K_1-1)V_1(N,n,\mathbf{K})$. The remaining terms of $H(N,n,\mathbf{K})$ are no longer divisible by (K_1-1). We then proceed to sum all the remaining terms that are divisible by (K_2-1), and get $(K_2-1)V_2(N,n,K_2,\dots,K_r)$ as the sum. Successively we sum the terms until we reach the last sum, namely, $(K_r-1)V_r(N,n,K_r)$.

Step 3. Divide (4.2) by $F(n,\mathbf{k})$ on both sides to get

$$\sum_{j=0}^{J}\frac{a_j(n)F(n-j,\mathbf{k})}{F(n,\mathbf{k})}=\sum_{i=1}^{r}\Bigg(R_i(n,k_1,k_2,\dots,k_r) - \frac{R_i(n,k_1,\dots,k_i-1,\dots,k_r)F(n,k_1,\dots,k_i-1,\dots,k_r)}{F(n,\mathbf{k})}\Bigg). \tag{4.3}$$

Step 4. Into (4.3), substitute for $R_i(n,\mathbf{k})$

$$\sum_{\substack{0\le j\le J\\ 0\le \mathbf{i}\le(0,\dots,0,I_i-1,I_{i+1},\dots,I_r)}}\frac{d(\mathbf{i},j,n)F(n-j,\mathbf{k}-\mathbf{i})}{F(n,\mathbf{k})},$$

where d's are polynomials in n only.

Step 5. Compute a common denominator for $R_i(n,\mathbf{k})$, i.e.,

$$D_{R_i}(n,\mathbf{k})=P(n,\mathbf{k})\prod_{s=1}^{p}(a_sn+\mathbf{b}_s\cdot\mathbf{k}+c_s)^{\overline{(a_s)^+J+(I_i-1)(b_{is})^+ +\sum_{i<t\le r}I_t(b_{ts})^+}}$$
$$\times\prod_{s=1}^{q}(u_sn+\mathbf{v}_s\cdot\mathbf{k}+w_s+1)^{\overline{(-u_s)^+J+(I_i-1)(-v_{is})^+ +\sum_{i<t\le r}I_t(-v_{ts})^+}}.$$

Step 6. Calculate the degree in $\mathbf{k}$ of the numerator polynomial $N_{R_i}(n,\mathbf{k})$ over the denominator polynomial $D_{R_i}(n,\mathbf{k})$.

$$\mathcal{N}_i:=\deg_{\mathbf{k}}P(n,\mathbf{k})+(I_i-1)(\mathcal{B}_i+(V_i-B_i)^+)+J(\mathcal{A}+(U-A)^+)+\sum_{i<t\le r}I_t(\mathcal{B}_t+(V_t-B_t)^+),$$

where (as in Chapter 3)

$$U:=\sum_{\substack{s\\ \mathbf{v}_s\ne 0}}u_s,\quad V_l:=\sum_{s}v_{ls},\quad A:=\sum_{\substack{s\\ \mathbf{b}_s\ne 0}}a_s,\quad B_l:=\sum_{s}b_{ls},$$

and

$$\mathcal{A} := \sum_{\substack{s \\ \mathbf{b}_s \neq 0}} (a_s)^+ + \sum_{\substack{s \\ \mathbf{v}_s \neq 0}} (-u_s)^+, \quad \mathcal{B}_l := \sum_s (b_{ls})^+ + \sum_s (-v_{ls})^+.$$

Step 7. Conclude that $R_i(n, \mathbf{k})$ has the form

$$\sum_{\substack{0 \leq \mathbf{e} \leq (\mathcal{N}_i, \mathcal{N}_i, \ldots, \mathcal{N}_i) \\ \mathbf{e} \cdot \mathbf{1} \leq \mathcal{N}_i}} \frac{c_i(\mathbf{e}, n)\mathbf{k}^{\mathbf{e}}}{D_{R_i}(n, \mathbf{k})}.$$

Step 8. Substitute R_i's into (4.3), and collect all terms to one side of the equal sign.

Step 9. Find a common denominator for the resulting expression and make the coefficients of each monomial in $\mathbf{k}$ zero.

Step 10. Solve the resulting system of homogeneous equations for the a_j's and c_i's. Again we are guaranteed that a solution exists from Theorem 4.1 in [WZ3].

CHAPTER V

SOME HYPERGEOMETRIC IDENTITIES ARE ALMOST TRIVIAL

Zeilberger [Z1] once claimed, *All binomial identities are verifiable.* His reasoning went as follows. Let $F(n, \mathbf{k})$ be a hypergeometric term. Then $\sum_{\mathbf{k}} F(n, \mathbf{k})$ satisfies a recurrence with polynomial-in-n coefficients. If we want to prove that $\sum_{\mathbf{k}} F(n, \mathbf{k}) = f(n)$ for some hypergeometric term $f(n)$, then we need to check that $f(n)$ satisfies the same recurrence as $\sum_{\mathbf{k}} F(n, \mathbf{k})$, and $f(n)$ agrees with $\sum_{\mathbf{k}} F(n, \mathbf{k})$ for all relevant n's less than or equal to the sum of the order of the recurrence and the highest integer zero of the leading coefficient of the recurrence. Since there are only a finite number of cases to check, it is sufficient to verify $\sum_{\mathbf{k}} F(n, \mathbf{k}) = f(n)$ with a pocket calculator.

However, no a priori bounds for the recurrence or the highest integer root of the leading coefficient were known at the time of the paper. In order to obtain an effective algorithm for proving a hypergeometric identity, we make use of the mathematical tools developed in [WZ3]. Using the terminology of [WZ3], we consider only admissible proper-hypergeometric terms to obtain a recurrence of the sum from that of the summand. In this chapter, we consider the case with one summation index, and calculate an a priori bound for the number of n's for which the hypergeometric identity $\sum_k F(n, k) = f(n)$ should be checked to establish the truth of the identity. These a priori bounds are quite astronomical in size, but they are finite, and pre-computable. (See the end of this chapter for examples of the sizes of the bounds.)

Main Theorem. *Let*

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k$$

be an admissible proper-hypergeometric term, and $P(n,k)$ *be a polynomial with coefficients in* $\mathbb{Z}$. *Let*

$$x := \max_s\{|a_s|, |b_s|, |c_s|, |u_s|, |v_s|, |w_s|\},$$

$$y := \max\{p, q\},$$

$$z := \max_{0\le i,j} \left|[n^j k^i]P(n,k)\right|,$$

$$d := 1 + \max\{\deg_k P(n,k), \deg_n P(n,k)\},$$

and let n_0 *be a given integer. If* $\sum_k F(n,k) = 1$ *for*

$$n_0 \le n \le (3xy)^{3(d+1)^2(2xy)^6} d^{5(d+1)(2xy)^3} z^{(d+1)(2xy)^3},$$

then $\sum_k F(n,k) = 1$ *for* <u>*all*</u> $n \ge n_0$.

Note that if we would like to prove that $\sum_k F(n,k) = f(n)$ for some hypergeometric term $f(n)$, then dividing both sides by $f(n)$, we get $\sum_k F(n,k)/f(n) = 1$. What remains is to check whether $F(n,k)/f(n)$ is an admissible proper-hypergeometric term before we can apply the theorem.

We first state and prove the following theorem which contains a much better bound, but the bound is in an even more complicated form (5.13).

Theorem 5.1. *Let*

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k \tag{5.1}$$

be an admissible proper-hypergeometric term, and $P(n,k)$ be a polynomial with coefficients in $\mathbb{Q}$. Then, given n_0, there exists an effectively computable positive integer n_1 such that if $\sum_k F(n,k) = 1$ for all $n_0 \leq n < n_1$, then $\sum_k F(n,k) = 1$ for all $n \geq n_0$. (See (5.13) for n_1.)

First we claim that it suffices to prove Theorem 5.1 for those polynomials $P(n,k)$ with integer coefficients. For if $P(n,k)$ is a polynomial with coefficients in $\mathbb{Q}$, then $P(n,k) = \tilde{P}(n,k)/d_p$ where $\tilde{P}(n,k)$ is a polynomial with integer coefficients and d_p is the least common multiple of the denominators of the coefficients of $P(n,k)$. But in order to prove that

$$\sum_k F(n,k) = \sum_k P(n,k) \frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!} \xi^k = 1,$$

it is equivalent to prove that

$$\sum_k \tilde{P}(n,k) \frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!} \xi^k = d_p.$$

5.1 Examples

In this section, we show some examples of hypergeometric identities whose leading coefficients, $a_0(n)$, in the recurrence equations vanish at embarrassing places of n, namely those n where the hypergeometric identities hold.

Why does a vanishing leading coefficient, $a_0(n)$, in the recurrence relation present a problem? Given a proper-hypergeometric term $F(n,k)$ and an integer n_0, suppose we want to prove $\sum_k F(n,k) = 1$ for all $n \geq n_0$. We know that $\sum_k F(n,k)$ satisfies some k-free recurrence of the form

$$a_0(n) \sum_k F(n,k) + a_1(n) \sum_k F(n-1,k) + \cdots + a_J(n) \sum_k F(n-J,k) = 0, \qquad n \geq n_0,$$

for some positive integer $J \le \sum_s |b_s| + \sum_s |v_s|$. (See [WZ3] Theorem 3.1, or a sharper bound in Chapter 1.) It is easy to check if 1 (the RHS) also satisfies the same recurrence. That is, we need to check if

$$a_0(n) + a_1(n) + \cdots + a_J(n) = 0 \tag{5.2}$$

for all n. To do so, we use the fact that if a polynomial, P, of degree d has $d+1$ zeros, then $P = 0$. Thus it suffices to check that (5.2) is true for $1 + \max_{j \in [J]_0} \deg a_j(n)$ different values of n. In addition, if $a_0(n) \neq 0$ for all $n \ge n_0$, then we can divide by $a_0(n)$ to get

$$\begin{aligned}(5.3)\quad &\sum_k F(n,k) \\ &= -\frac{a_1(n)\sum_k F(n-1,k) + a_2(n)\sum_k F(n-2,k) + \cdots + a_J(n)\sum_k F(n-J,k)}{a_0(n)}.\end{aligned}$$

Now, (5.3) is a Jth order recurrence in n for $\sum_k F(n,k)$, so if $\sum_k F(n,k) = 1$ for $n_0 \le n \le \max\{n_0 + J - 1, n_0 + \max_{j \in [J]_0} \deg a_j(n)\}$, then it follows (by induction and using (5.2)) that $\sum_k F(n,k) = 1$ for all $n \ge n_0$.

However, if $a_0(n)$ vanishes for some $n \ge n_0$, then (5.2) and (5.3) does not hold at that particular n. In order to use the recurrence relation to establish the identity, we need to know an integer $n_a \ge J$ such that $a_0(n)$ does not vanish for all $n \ge n_a$, and check individually that $\sum_k F(n,k) = 1$ for all $n \in \{n_0, n_0 + 1, \ldots, \max\{n_a - 1 + J, n_0 + \max_{j \in [J]_0} \deg a_j(n)\}\}$. □

The first example shows that n_1 in Theorem 5.1 depends on the coefficients of $P(n,k)$.

Example 1. Suppose we wanted to evaluate the sums

$$s_n = \sum_k (k^2 - 9k + 4)\binom{n}{k}, \qquad n \ge 0.$$

We show that s_n satisfies the following recurrence relation,

$$(n-16)(n-1)s_{n+1} - 2n(n-15)s_n = 0.$$

Let

$$F(n,k) = (k^2 - 9k + 4)\binom{n}{k}, \quad \text{and} \quad f_n(x) = \sum_k F(n,k)x^k.$$

Then

$$\begin{aligned} f_n(x) &= \sum_k (k^2 - 9k + 4)\binom{n}{k} x^k \\ &= (1+x)^{n-2}\left(n(n-1)x^2 + nx(1+x) - 9nx(1+x) + 4(1+x)^2\right). \end{aligned}$$

Therefore $s_n = f_n(1) = 2^{n-2}(n^2 - 17n + 16)$, and $s_{n+1} = f_{n+1}(1) = 2^{n-1}((n+1)^2 - 17(n+1) + 16)$. From the expressions of s_n and s_{n+1}, we conclude that our desired sums s_n satisfy the recurrence

$$(n-16)(n-1)s_{n+1} - 2n(n-15)s_n = 0,$$

or equivalently,

$$s_{n+1} = \frac{2n(n-15)}{(n-16)(n-1)} s_n \qquad \text{for} \quad n > 16.$$

Notice that the recurrence relation can be used to calculate s_n successively only if $n > 16$, because the leading coefficient vanishes at $n = 16$ and $n = 1$. Thus, if we check individually that $f_n(1) = \sum_k F(n,k)$ for $n = 0, 1, \ldots, 16, 17$, then we can use the recurrence relation to calculate s_n for $n \geq 18$. In this example, n_1 of Theorem 5.1 is 18. □

Example 2. In this example, we show that n_1 depends on the coefficients of $P(n,k)$ and might be arbitrarily large.

Fix a large $n_1 \in \mathbb{N}$. We consider more generally,

$$F(n,k) = (ak^2 + bk + c)\binom{n}{k} \quad \text{and} \quad f_n(x) = \sum_k F(n,k)x^k.$$

We take the given n_1 and find a, b, c in $\mathbb{Q}$ such that the sum

$$s_n = \sum_k F(n,k) = f_n(1)$$

satisfies a recurrence relation whose leading coefficient, $a_0(n)$ vanishes at $n_1 - 2$.

After a similar calculation as the one in Example 1, we obtain the recurrence

$$(an^2 + n(a+2b) + 4c)s_{n+1} - 2(a(n+1)^2 + (n+1)(a+2b) + 4c)s_n \doteq 0.$$

The coefficient of s_{n+1} vanishes at

$$n = \frac{-(a+2b) \pm \sqrt{(a+2b)^2 - 16ac}}{2a}.$$

If the discriminant is greater than 0, then the larger of the two roots is

$$n^* = \frac{-(a+2b) + \sqrt{(a+2b)^2 - 16ac}}{2a}.$$

If we find some positive integers α and β such that $\gcd(\alpha,\beta) = 1$ and $\alpha > \beta$ satisfying

$$(a+2b)^2 = (\alpha+\beta)^2$$

$$16ac = 4\alpha\beta$$

simultaneously, then

$$\sqrt{(a+2b)^2 - 16ac} = \sqrt{(\alpha-\beta)^2} = \alpha - \beta.$$

In this case,

$$n^* = \frac{(\alpha+\beta) + (\alpha-\beta)}{2a} = \frac{\alpha}{a}.$$

Since we want $n^* = n_1 - 2$, we can take $a = 1$, $\alpha = n_1 - 2$, $\beta = 1$. It follows that

$$b = \frac{-(\alpha+\beta) - 1}{2}, \quad c = \frac{\alpha\beta}{4},$$

and $n^* = \alpha = n_1 - 2$. $\square$

Example 3. This example shows that even if the summand $F(n,k)$ consists only of factorial parts, and does not have a polynomial part, then it may happen that the leading coefficient of the recurrence satisfied by the sum vanishes at some positive integers n where $\sum_k F(n,k)$ is summable for those integers.

Fix a large $n_1 \in \mathbf{N}$, and take the summand in Saalschütz' identity,

$$F(n,k) = \frac{(a+k-1)!\,(b+k-1)!\,n!\,(-a-b+c+n-1-k)!}{k!\,(n-k)!\,(c+k-1)!}.$$

Then Saalschütz says that

$$\sum_k F(n,k) = \frac{(c-a)^{\overline{n}}(c-b)^{\overline{n}}}{c^{\overline{n}}(c-a-b)^{\overline{n}}} =: f_n.$$

It is easy to check that $F(n,k)/f_n$ satisfies the hypothesis of Theorem 5.1. The recurrence for f_n is

$$(n+c)(n+c-a-b)f_{n+1} - (n+c-a)(n+c-b)f_n = 0.$$

It suffices to pick a, b in $\mathbb{Z}^-$ and $c := -(n_1 - 2)$. $\square$

5.2 Two approximation lemmas

Notation. We use $[n]$ to denote $\{1, 2, \ldots, n\}$, $[n]_0$ to denote $\{0\} \cup [n]$, and $[x^m y^n]P(x,y)$ to denote the coefficient of $x^m y^n$ in $P(x,y)$. We use $P(n,k) \preccurlyeq Q(n,k)$ to mean that for all pairs of integers, (m,l), $|[n^m k^l]P(n,k)| \le |[n^m k^l]Q(n,k)|$.

We need the following lemmas for the proof of Theorem 5.1.

Lemma 5.2. *Let $P(n,k)$ be a polynomial in n and k with integer coefficients, and let*

$$\mu = \max_{l \in [E]_0, m \in [D]_0} |[n^m k^l]P(n,k)|, \quad D = \deg_n P(n,k), \quad \text{and} \quad E = \deg_k P(n,k).$$

Then for every positive integer J,

$$\max_{l\in[E]_0,m\in[D]_0,j\in[J]_0} |[n^m k^l]P(n-j,k)| \leq (1+J)^D \mu.$$

Proof. Suppose $P(n,k)=\sum_{l=0}^{E}\sum_{m=0}^{D} t_{lm}k^l n^m$. Then, $|[k^l n^m]P(n-j,k)|$ is

$$\begin{aligned}\left|\sum_{i=0}^{D-m}(-1)^i t_{l,m+i}\binom{m+i}{m} j^i\right| &\leq \sum_{i=0}^{D}\binom{D}{i} J^i \max_{l\in[E]_0,m\in[D]_0} |[n^m k^l]P(n,k)| \\ &= (1+J)^D\mu\end{aligned}$$

for all $j\in[J]\cup\{0\}$, $l\in[E]_0$ and $m\in[D]_0$. □

Lemma 5.3. *Let $Q(n,k)=\prod_{s=1}^{q}(a_s n+b_s k+c_s)$, where a_s,b_s,c_s are integers. Then*

$$\max_{m,l\in[q]_0} |[n^m k^l]Q(n,k)| < 3^q \prod_{s\in[q]} \max\{|a_s|,|b_s|,|c_s|\}.$$

Proof. We know that

$$\begin{aligned}Q(n,k) &= \prod_{s=1}^{q}(a_s n+b_s k+c_s) \\ &\preccurlyeq (n+k+1)^q \prod_{s\in[q]} \max\{|a_s|,|b_s|,|c_s|\}.\end{aligned}$$

Since the absolute value of the largest coefficient of $(n+k+1)^q$ is the trinomial coefficient $\binom{q}{\lfloor\frac{q}{3}\rfloor,\lceil\frac{q}{3}\rceil,q-\lfloor\frac{q}{3}\rfloor-\lceil\frac{q}{3}\rceil}$, we get that

$$\begin{aligned}\max_{m,l\in[q]_0} |[n^m k^l]Q(n,k)| &\leq \binom{q}{\lfloor\frac{q}{3}\rfloor,\lceil\frac{q}{3}\rceil,q-\lfloor\frac{q}{3}\rfloor-\lceil\frac{q}{3}\rceil} \prod_{s\in[q]} \max\{|a_s|,|b_s|,|c_s|\} \\ &< 3^q \prod_{s\in[q]} \max\{|a_s|,|b_s|,|c_s|\}. \quad \square\end{aligned}$$

5.3 Solving a homogeneous symbolic linear system

Definition. Let M be an $l \times m$ matrix over the field of rational functions over $\mathbb{Q}$. Define the *generic rank* of M to be the number of non-zero rows in the reduced row-echelon form of M. Since row rank (M) = column rank (M) = rank (M) from [H, p. 337], the generic rank is the classical definition of the rank of a matrix over a division ring. Henceforth we use rank to mean the generic rank.

In this section, we consider a special class of matrices M, $l \times m$ such that $l < m$, and M_{ij}, the entries of M, are polynomials in x with integer coefficients. Since M is a subset of the matrices over the field of rational functions, the rank of M is well-defined.

Let $\mathbf{x}$ be an $m \times 1$ vector with indeterminate polynomial entries, $\{a_n(x)\}_1^m$, with integer coefficients. The problem is to solve for $\mathbf{x}$ in $M\mathbf{x} = \mathbf{0}$, for some M of non-zero rank. After obtaining a solution for $\mathbf{x}$, we estimate the degree and the largest coefficient of $a_n(x)$, for $n \in [m]$.

The following is a procedure for finding $\mathbf{x}$. Let $M\mathbf{x} = \mathbf{0}$ be a system of homogeneous linear equations such that

(1) M is $l \times m$, $l < m$,

(2) entries of M are polynomials in x with integer coefficients,

(3) M has rank $\rho > 0$,

(4) $\mathbf{x}^t = (a_1(x), \ldots, a_m(x))$,

(5) wlog, assume $a_1(x)$ is not identically zero.

Then $\mathbf{x}$ can be obtained from the following procedure.

Step A. By renumbering the unknowns, if necessary, and permuting the columns of M, we can arrange that the first ρ columns of M have rank ρ.

Step B. Interchange the rows of the resulting matrix from (1) above to make the $\rho \times \rho$ upper left hand corner of M, called M', a square matrix of rank ρ.

Step C. Set all but the first ρ variables in the new $\mathbf{x}$ to 1.

Step D. What remains is a system of ρ inhomogeneous linear equations in $x_1, x_2, \ldots, x_\rho$, say $M'\mathbf{x}' = \mathbf{y}'$. We note that $\mathbf{y}' \neq \mathbf{0}$. For if $\mathbf{y}' = \mathbf{0}$, and M' is of full rank, then the only solution to $M'\mathbf{x}' = \mathbf{0}$ is the zero solution. But $\mathbf{x}'$ has $a_1(x)$ as its first member, and $a_1(x)$ is assumed to be non-zero.

Step E. Use Cramer's rule to find the unknowns $\mathbf{x}'$, namely:

$$\text{the } n\text{th entry of } \mathbf{x}' = \frac{\det M'_n}{\det M'} \quad (n = 1, \ldots, \rho).$$

In particular,

$$a_1(x) = \frac{\det M'_1}{\det M'}.$$

Step F. To make the solution for $\mathbf{x}$ a polynomial solution, we multiply $\mathbf{x}$ by $\det M'$. Since M' is obtained from M by interchanging rows and columns, the entries of M' are still polynomials with integer coefficients. Therefore $\det M'$ is a polynomial over $\mathbb{Z}$. Similarly, each M'_n has entries over $\mathbb{Z}[x]$, for entries of $\mathbf{y}'$ are sums of some entries in M. Thus the complete solution vector is

$$\text{the new } \mathbf{x} = (\det M'_1, \ldots, \det M'_\rho, \det M', \ldots, \det M')^t.$$

5.4 Sufficient conditions for a polynomial not to vanish

Given $\{a_n(x)\}_1^m$ from the end of §5.3, we find in this section upper bounds for the degrees and the largest coefficient of the $\{a_n(x)\}_1^m$. With these bounds, we apply

Proposition 5.4. *Let $a(x) \in \mathbb{Z}[x]$, let $d = \deg a$, and let m be $\max_{i\in[d]_0} |[x^i]a(x)|$. Then $a(x) \neq 0$ for all $x > md$.*

Proof. Let

$$a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_dx^d, \quad a_d \neq 0, \quad (\text{all } a_i \in \mathbb{Z}).$$

Then for sufficiently large x,

$$\begin{aligned}|a(x)| &\geq x^d|a_d| - |x^{d-1}a_{d-1} + \cdots + a_0| \\ &\geq x^d|a_d| - x^{d-1}d \cdot \max\{|a_0|, |a_1|, \ldots, |a_{d-1}|\} \\ &= x^{d-1}(x|a_d| - d \cdot \max_{j\in[d-1]_0} \{|a_j|\}) > 0,\end{aligned}$$

if

$$x > \frac{d \max_{j\in[d-1]_0} |a_j|}{|a_d|}.$$

Since $a_d \neq 0$ and $a_d \in \mathbb{Z}$, the latter surely holds if $x > md$. □

We estimate the degree of $a_n(x)$, $n \in [m]$, from $\det M'$ and $\det M'_n$, the polynomial solutions for **x**.

Let

$$\mu := \max\{\deg M_{ij}(x) \mid i \in [l], j \in [m]\},$$

then $\deg\det M'(x) \leq \rho\mu$ for the following reason. Since M' is obtained from M by interchanging rows and columns, $\deg M'_{ij} \leq \mu$, for $i, j \in [\rho]$. We conclude that $\deg(\det M'(x)) \leq \rho\mu$ because $\operatorname{rank}(M') = \rho$. Similarly, $\deg(\det M'_n(x)) \leq \rho\mu$ because M'_n is obtained from M by interchanging rows and columns of M, and by adding some columns of M together to make $\mathbf{y}'$, then replacing the nth column of M' by $\mathbf{y}'$ to get M'_n.

Next we estimate the heights of the $a_n(x)$'s, i.e., $\max_{k\in[\rho\mu]} |[x^k]a_n(x)|$ for $n \in [m]$. Let

$$c := \max\left\{ \left|[x^k]M_{ij}(x)\right| \mid i \in [l], j \in [m], k \in [\mu] \right\},$$

then

$$\max_{i,j,k} |[x^k]M'_{ij}(x)| \leq c \quad \text{and} \quad \max_{i,j,k} |[x^k](M'_n)_{ij}(x)| \leq (m-\rho)c,$$

again from the ways M' and M'_n are obtained from M. With an upper bound for the maximum coefficient of M', we estimate $\max_k |[x^k] \det M'(x)|$ using the definition of the determinant. By the definition of the determinant, we have

$$\begin{aligned}
\det M' &= \sum_{\sigma\in S_\rho} \operatorname{sgn}(\sigma) e_{1\sigma(1)} e_{2\sigma(2)} \cdots e_{\rho\sigma(\rho)} \\
&\preccurlyeq \sum_{\sigma\in S_\rho} |e_{1\sigma(1)} e_{2\sigma(2)} \cdots e_{\rho\sigma(\rho)}| \\
&\preccurlyeq \rho!\, c^\rho (x^\mu + x^{\mu-1} + \cdots + x + 1)^\rho.
\end{aligned}$$

Thus

$$\max_k \left|[x^k] \det M'\right| \leq \rho!\, c^\rho (\mu+1)^\rho.$$

Similarly,

$$\max_k \left|[x^k] \det M'_n\right| \leq \rho!\, ((m-\rho)c)^\rho (\mu+1)^\rho.$$

5.5 The leading coefficient, $a_0(n)$, of the recurrence

We estimate the degree of the leading coefficient, $a_0(n)$, in the recurrence of $F(n,k)$ as a polynomial in n, and n_a, the positive integer with the property that for all $n \geq n_a$, $a_0(n) \neq 0$. The plan for achieving this goal consists of the following four stages:

Stage 1. Take a given admissible proper-hypergeometric term $F(n,k)$, and use Theorem 3.2A of [WZ3] to say that $F(n,k)$ satisfies a recurrence of the form:

$$a_0(n)F(n,k) + a_1(n)F(n-1,k) + \cdots + a_J(n)F(n-J,k) = G(n,k) - G(n,k-1), \tag{5.4}$$

where the $a_j(n)$'s are unknown polynomials in n. Divide (5.4) by $F(n,k)$ and put the resulting sum of rational functions over a common denominator.

Stage 2. Equate the coefficient of each power of k in the common numerator to 0, and solve the resulting homogeneous linear equations for the unknowns $a_j(n)$'s and $c_i(n)$'s (see (5.5) for $c_i(n)$'s) by Cramer's rule for $a_0(n)$ only, in the form

$$a_0(n) = \frac{\det M_1'}{\det M'}.$$

(See §5.3 Step E.)

Stage 3. Observe that $a_0(n) = 0$ exactly when $\det M_1' = 0$. Therefore, we express $\det M_1'$ as a polynomial in n, and obtain an upper bound for the degree of $\det M_1'$ (see §5.6 formula (5.11)) and the largest coefficient of $\det M_1'$ (see §5.6 formula (5.12)).

Stage 4. Use the simple fact that if $a(x)$ is a polynomial over $\mathbb{Z}$, d is the degree of $a(x)$ and m is $\max_{i\in[d]_0}|[x^i]a(x)|$, then $a(x) \neq 0$ for all $x > md$. (See Proposition 5.4 in §5.4.) Thus we use the estimates in Stage 3 to obtain an n_a such that for all $n > n_a$, $a_0(n) \neq 0$.

We now proceed to do Stage 1 of the plan in detail. Let an admissible proper-hypergeometric term $F(n,k)$ be given such that $P(n,k)$ in $F(n,k)$ has integer coefficients. Recall that

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k.$$

Then Theorem 3.2A of [WZ3] guarantees us the existence of polynomials $a_0(n)$, $a_1(n), \ldots,$ $a_J(n)$, not all zero, an integer, $J \leq \sum_s |b_s| + \sum_s |v_s|$, and a function $G(n,k)$ such that

$G(n,k) = R(n,k)F(n,k)$ for some rational function R and such that

$$(5.4)\quad a_0(n)F(n,k) + a_1(n)F(n-1,k) + \cdots + a_J(n)F(n-J,k) = G(n,k) - G(n,k-1).$$

Without loss of generality, assume $a_0(n)$ is not identically zero. From Chapter 2, we may assume that $R(n,k)$ has the form

$$\frac{\sum_{i=0}^{\mathcal{N}} c_i(n)k^i}{D_R(n,k)}$$

for some polynomials, $c_i(n)$ $(i \in [\mathcal{N}]_0)$, where

$$\mathcal{N} = \deg_k P(n,k) + J(\mathcal{A} + (U-A)^+) + (I-1)(\mathcal{B} + (V-B)^+).$$

(See Theorem 1.4 for the definitions of $\mathcal{A}$, $\mathcal{B}$, A, B, U and V.) Dividing both sides of (5.4) by $F(n,k)$, we get

$$(5.5)\quad a_0(n) + a_1(n)\frac{F(n-1,k)}{F(n,k)} + \cdots + a_J(n)\frac{F(n-J,k)}{F(n,k)}$$
$$= \frac{\sum_{i=0}^{\mathcal{N}} c_i(n)k^i}{D_R(n,k)} - \frac{\sum_{i=0}^{\mathcal{N}} c_i(n)(k-1)^i}{D_R(n,k-1)}\frac{F(n,k-1)}{F(n,k)}.$$

Putting (5.5) over a common denominator $D(n,k)$, we find that we can take

$$D(n,k) := P(n,k)\prod_{s=1}^{p}(a_sn+b_sk+c_s)^{\overline{(a_s)^+J+(b_s)^+I}}\prod_{s=1}^{q}(u_sn+v_sk+w_s+1)^{\overline{(-u_s)^+J+(-v_s)^+I}}$$
$$\times\prod_{s=1}^{p}(a_sn+b_sk+c_s-b_s+1)^{\overline{(b_s)^+}}\prod_{s=1}^{q}(u_sn+v_sk+w_s-v_s)^{\overline{(-v_s)^+}}.$$

Next we collect all terms of (5.5) to the left side to get

$$(5.6)\qquad \frac{L_0 + L_1 + \cdots + L_J - R_1 + R_2}{D(n,k)} = 0,$$

where for $0 \le j \le J$,

$$
\begin{aligned}
L_j(n,k) := a_j(n)P(n-j,k) & \prod_s (a_s n + b_s k + c_s + 1)^{\overline{j(-a_s)^+}} \\
& \times \prod_s (a_s n + b_s k + c_s - (a_s)^+ J + 1)^{\overline{(J-j)(a_s)^+}} \\
& \times \prod_s (a_s n + b_s k + c_s - b_s + 1)^{\overline{(b_s)^+}} \\
& \times \prod_s (a_s n + b_s k + c_s - (a_s)^+ J)^{\underline{I(b_s)^+}} \\
& \times \prod_s (u_s n + v_s k + w_s + (-u_s)^+ J)^{\underline{(J-j)(-u_s)^+}} \\
& \times \prod_s (u_s n + v_s k + w_s)^{\underline{j(u_s)^+}} \\
& \times \prod_s (u_s n + v_s k + w_s + (-u_s)^+ J + 1)^{\overline{I(-v_s)^+}} \\
& \times \prod_s (u_s n + v_s k + w_s - v_s)^{\underline{(-v_s)^+}},
\end{aligned}
$$

and

$$
\begin{aligned}
R_1(n,k) := \left(\sum_{i=0}^{\mathcal{N}} c_i(n) k^i \right) & \prod_s (a_s n + b_s k + c_s - (a_s)^+ J - (b_s)^+ I + 1)^{\overline{(b_s)^+}} \\
& \times \prod_s (a_s n + b_s k + c_s - b_s + 1)^{\overline{(b_s)^+}} \\
& \times \prod_s (u_s n + v_s k + w_s + (-u_s)^+ J + (-v_s)^+ I)^{\underline{(-v_s)^+}} \\
& \times \prod_s (u_s n + v_s k + w_s - v_s)^{\underline{(-v_s)^+}},
\end{aligned}
$$

and

$$
\begin{aligned}
R_2(n,k) := & \left(\sum_{j=0}^{\mathcal{N}} (-1)^j \sum_{i=j}^{\mathcal{N}} c_i(n) \binom{i}{j} k^{i-j} \right) \\
& \times \prod_s (a_s n + b_s k + c_s + 1)^{\overline{(-b_s)^+}} \prod_s (a_s n + b_s k + c_s)^{\underline{(b_s)^+}} \\
& \times \prod_s (u_s n + v_s k + w_s)^{\underline{(v_s)^+}} \prod_s (u_s n + v_s k + w_s + 1)^{\overline{(-v_s)^+}}.
\end{aligned}
$$

We now do Stage 2 of the plan. Our goal is to find an expression for $a_0(n)$. To solve for the unknown polynomials in n,

$$a_0, a_1, \ldots a_J, c_0, c_1, \ldots, c_{\mathcal{N}},$$

we expand the terms of (5.6), and collect like powers of k. Since LHS of (5.6) is zero, the coefficients of k^l must be identically zero. This yields a system of linear homogeneous equations. We then express the system as $M\mathbf{x} = \mathbf{0}$, $\mathbf{x}^t = (a_0, a_1, \ldots, a_J, c_0, c_1, \ldots, c_{\mathcal{N}})$, and the ith row of M corresponds to the coefficients of k^{i-1} in the common numerator of (5.6). We are now set to apply the procedure in §5.3 for finding a polynomial $a_0(n)$ using Cramer's rule.

Stage 3 of the plan consists of three steps. First, we find an upper bound for the maximum degree of the entries of M. (See Lemma 5.5.) Second, we find an upper bound for the largest coefficient of the entries of M. (See Lemma 5.6.) With these estimates for the entries of M, we find upper bounds for the degree and the largest coefficient of det $M_1^\prime$ ($= a_0(n)$) using §5.4.

Step 1. An upper bound for the maximum degree of all entries of M regarded as a polynomial in n.

Lemma 5.5. *Let*

$$\mu_1 := \deg_n P(n,k) + (I+1)\tilde{\mathfrak{B}} + J(\tilde{\mathcal{A}} + (\tilde{U} - \tilde{A})^+),$$

where

$$\tilde{\mathfrak{B}} = \sum_{s:a_s \neq 0} b_s^+ + \sum_{s:u_s \neq 0} (-v_s)^+, \quad \tilde{\mathcal{A}} = \sum_{s\in[p]} a_s^+ + \sum_{s\in[q]} (-u_s)^+,$$
$$\tilde{U} = \sum_{s\in[q]} u_s, \qquad \tilde{A} = \sum_{s\in[p]} a_s,$$

and let

$$\mu_2 := \max\left\{\sum_{a_s\neq 0}|b_s| + \sum_{u_s\neq 0}|v_s|, \quad 2\sum_{a_s\neq 0} b_s^+ + 2\sum_{u_s\neq 0}(-v_s)^+\right\}.$$

Then

$$\max_{i,j}\deg M_{ij}(n) \leq \max\{\mu_1, \mu_2\}.$$

Proof. Let μ be the maximum degree of the entries of M. By the setup, M_{1j} is the coefficient of k^0, that is, $P(n-j,k)$ times the first elementary symmetric function when the product

$$\frac{L_j}{a_j(n)P(n-j,k)}$$

is viewed as a product of terms of the type $(b_s k + d_s(n))$, where $d_s(n)$ is a polynomial in n of degree 1 at most. Therefore, $\deg M_{ij} \leq \deg M_{1j}$ for $1 \leq j \leq J+1$. But

$$\deg M_{1j}(n) = \deg_n P(n,k) + (J+1-j)\left(\sum_s (a_s)^+ + \sum_s(-u_s)^+\right)$$
$$+ (I+1)\left(\sum_{a_s\neq 0}(b_s)^+ + \sum_{u_s\neq 0}(-v_s)^+\right) + (j-1)\left(\sum_s(-a_s)^+ + \sum_s(u_s)^+\right),$$

for $1 \leq j \leq J+1$. Thus

$$\max_j \deg M_{1j}(n) = \deg_n P(n,k) + (I+1)\tilde{\mathcal{B}} + J(\tilde{A} + (\tilde{U} - \tilde{A})^+) = \mu_1,$$

where

$$\tilde{\mathcal{B}} = \sum_{a_s\neq 0} b_s^+ + \sum_{u_s\neq 0}(-v_s)^+, \quad \tilde{A} = \sum a_s^+ + \sum(-u_s)^+,$$
$$\tilde{U} = \sum u_s, \qquad \tilde{A} = \sum a_s.$$

For $1 \leq i \leq \nu, J+2 \leq j \leq J+2+\mathcal{N}$, M_{ij} is the polynomial (in n) multiplied by $k^{i-1}c_{j-(J+2)}(n)$. To find $\deg M_{ij}$, we compute R_1 and R_2 to conclude that

$$\deg M_{ij} \leq \max\left\{\sum_{a_s\neq 0}|b_s| + \sum_{u_s\neq 0}|v_s|, \quad 2\sum_{a_s\neq 0} b_s^+ + 2\sum_{u_s\neq 0}(-v_s)^+\right\} = \mu_2.$$

Thus $\mu \leq \max\{\mu_1, \mu_2\}$. □

Step 2. An upper bound for $\max_{i,j,l} |[n^l]M_{i,j}(n)|$.

We note that M_{ij} is the polynomial in n multiplied either by $a_{j-1}(n)k^{i-1}$ for all $i \in [\nu]$ and $j \in [J+1]$, or by $c_{j-(J+2)}k^{i-1}$ for $i \in [\nu]$ and $j \geq J+2$. First, we compute an upper bound for $\max_{i,j,l} |[n^l]M_{i,j}(n)|$ for all i and $j \in [J+1]$. The parameters in c_s and w_s are fixed arbitrarily in this step.

Let

$$P(n,k) := \sum_{l=0}^{E} \sum_{m=0}^{D} t_{lm} n^m k^l,$$

$$\mu_3 := (1+J)^D \max\{|t_{lm}| : l \in [E]_0 \text{ and } m \in [D]_0\},$$

$$\begin{aligned}
\mu_4 := & \prod_{s:b_s>0} \prod_{i \in [b_s^+]} \max\{|a_s|, |b_s|, |c_s - b_s + i|\} \\
& \times \prod_{s:v_s<0} \prod_{i \in [(-v_s)^+]} \max\{|u_s|, |v_s|, |w_s - v_s + 1 - i|\} \\
& \times \prod_{s:b_s>0} \prod_{i \in [I(b_s^+)]} \max\{|a_s|, |b_s|, |c_s - J(a_s^+) + 1 - i|\} \\
& \times \prod_{s:v_s<0} \prod_{i \in [I(-v_s)^+]} \max\{|u_s|, |v_s|, |w_s + (-u_s)^+ J + i|\},
\end{aligned}$$

$$\begin{aligned}
\mu_5 := \max_{0 \leq j \leq J} \Bigg(& \prod_{s:a_s<0} \prod_{i \in [j(-a_s)^+]} \max\{|a_s|, |b_s|, |c_s + i|\} \\
& \times \prod_{s:a_s>0} \prod_{i \in [(J-j)a_s^+]} \max\{|a_s|, |b_s|, |c_s - Ja_s^+ + i|\} \\
& \times \prod_{s:u_s<0} \prod_{i \in [(J-j)(-u_s)^+]} \max\{|u_s|, |v_s|, |w_s + J(-u_s)^+ + 1 - i|\} \\
& \times \prod_{s:u_s>0} \prod_{i \in [j(u_s)^+]} \max\{|u_s|, |v_s|, |w_s + 1 - i|\} \Bigg),
\end{aligned}$$

and

$$e_1 := J(\tilde{A} + (\tilde{U} - \tilde{A})^+) + (I+1)(\sum_s b_s^+ + \sum_s (-v_s)^+).$$

We know from Lemma 5.3 that

$$\frac{L_j}{a_j} \preccurlyeq P(n-j,k)\mu_4\mu_5(n+k+1)^{e_1}.$$

Moreover, Lemma 5.2 states that

$$\max_{m,l} |[n^m k^l]P(n-j,k)| \leq \mu_3$$

for all $j \in [J]_0$. We conclude that the largest coefficient of M_{ij} for $i \in [\nu]$ and $j \in [J+1]$ is bounded above by

$$(D+1)(E+1)\mu_3\mu_4\mu_5 3^{e_1}.$$

Now we find an upper bound for the largest coefficient of $M_{i,j+J+2}$ for $i \in [\nu]$ and $j \geq 0$. As we observed before, $M_{i,j+J+2}$ is the polynomial in n multiplied by $c_j k^{i-1}$. By expanding R_1 and R_2, we get

$$M_{i,j+J+2} = -[k^{i-1-j}]\frac{R_1}{\sum_{l=1}^{N} c_l k^l} + \sum_{l=0}^{j}(-1)^l \binom{j}{l} [k^{i-1-j+l}] \frac{R_2}{\sum_j (-1)^j \sum_l c_l \binom{l}{j} k^{l-j}}.$$

Let

$$\begin{aligned}
\mu_6 := \prod_{s:b_s>0} \Bigg(\prod_{i\in[b_s^+]} & \max\{|a_s|,|b_s|,|c_s - Ja_s^+ - Ib_s^+ + i|\} \times \max\{|a_s|,|b_s|,|c_s - b_s^+ + i|\}\Bigg) \\
\times \prod_{s:v_s<0} \Bigg(\prod_{i\in[(-v_s)^+]} & \max\{|u_s|,|v_s|,|w_s - v_s + 1 - i|\} \\
& \times \max\{|u_s|,|v_s|,|w_s + J(-u_s)^+ + I(-v_s)^+ + 1 - i|\}\Bigg),
\end{aligned}$$

$$\begin{aligned}
\mu_7 := & \prod_{s:b_s<0} \prod_{i\in[(-b_s)^+]} \max\{|a_s|,|b_s|,|c_s + i|\} \prod_{s:b_s>0} \prod_{i\in[b_s^+]} \max\{|a_s|,|b_s|,|c_s + 1 - i|\} \\
& \times \prod_{s:v_s<0} \prod_{i\in[(-v_s)^+]} \max\{|u_s|,|v_s|,|w_s + i|\} \prod_{s:v_s>0} \prod_{i\in[v_s^+]} \max\{|u_s|,|v_s|,|w_s + 1 - i|\},
\end{aligned}$$

and

$$e_2 := 2\left(\sum_s b_s^+ + \sum_s (-v_s)^+\right),$$
$$e_3 := \sum_s |b_s| + \sum_s |v_s|.$$

By Lemma 5.3,

$$\max_{m,l}\left|[n^m k^l]\left(\frac{R_2}{\sum_{j\geq 0}(-1)^j \sum_{i\geq j} c_i \binom{i}{j} k^{i-j}}\right)\right| \quad \left(\text{resp.} \quad \max_{m,l}\left|[n^m k^l]\left(\frac{R_1}{\sum_i c_i k^i}\right)\right|\right)$$

is bounded above by

$$3^{e_3}\mu_7 \quad (\text{resp.} \quad 3^{e_2}\mu_6).$$

Thus, from the expression of M_{ij}, we conclude that the largest coefficient is bounded above by

$$2^{\mathcal{N}}3^{e_3}\mu_7 + 3^{e_2}\mu_6.$$

Let

$$\mu_8 := \max\left\{(D+1)(E+1)3^{e_1}\mu_3\mu_4\mu_5, \quad 2^{\mathcal{N}}3^{e_3}\mu_7 + 3^{e_2}\mu_6\right\}.$$

Then the largest coefficient of M_{ij} for $j \in [\mathcal{N}+J+2]$ and $i \in [\nu]$ is bounded above by μ_8. Thus we have

Lemma 5.6. *The absolute value of the largest coefficient of the entries of M is bounded by μ_8.*

Step 3. Upper bounds for $\deg\det M_1'$ *and* $\max_i |[n^i]\det M_1'|$.

We take the M' and M_1' obtained from §5.3, and perform the computation of §5.4 to get

$$\deg\det M_1' \leq \operatorname{rank}(M') \times \max\{\mu_1, \mu_2\},$$

where μ_1 and μ_2 are from Lemma 5.5, and

$$\max_i |[n^i]\det M_1^i| \le \rho!\left((J+\mathcal{N}+2-\rho)\mu_8(1+\max\{\mu_1,\mu_2\})\right)^\rho,$$

where ρ :=rank(M').

Stage 4 is the application of Proposition 5.4 using the bounds we just calculated in Step 3 above. We have now completed the computation needed for the proof of Theorem 5.1.

5.6 Proof of Theorem 5.1

Proof of Theorem 5.1. Let $(n,k) \in \mathbb{Z}^2$ be a point at which $F(n,k) \neq 0$, and such that $F(n-j,k-i)$ is well-defined for all $i \in [I]_0$ and $j \in [J]_0$, where I and J are some integers bounded above by the expressions found in [WZ3, Theorem 3.1] or a sharper bound from Theorem 1.4. By Theorem 3.2A of [WZ3], there exist polynomials $a_0(n), a_1(n), \ldots, a_J(n)$ not all identically zero, and a function $G(n,k)$ such that $G(n,k) = R(n,k)F(n,k)$ for some rational function R and such that $a_0(n)$ is not identically 0 and

$$a_0(n)F(n,k) + a_1(n)F(n-1,k) + \cdots + a_J(n)F(n-J,k) = G(n,k) - G(n,k-1). \tag{5.7}$$

From Step 3 of Chapter 2, we know that for

$$\mathcal{N} := \deg_k P(n,k) + J(\mathcal{A} + (U-A)^+) + (I-1)(\mathcal{B} + (V-B)^+),$$

where $A, B, U, V, \mathcal{A}, \mathcal{B}$ are defined in Theorem 1.4, the rational function $R(n,k)$ in (5.7) assumes the form

$$\frac{\sum_{i=0}^{\mathcal{N}} c_i(n)k^i}{D_R(n,k)}.$$

Substituting the expression for $R(n,k)$ into (5.7), then dividing both sides by $F(n,k)$, we get an equation of rational functions

$$a_0(n) + a_1(n)\frac{F(n-1,k)}{F(n,k)} + \cdots + a_J(n)\frac{F(n-J,k)}{F(n,k)} = R(n,k) - \frac{R(n,k-1)F(n,k-1)}{F(n,k)}. \tag{5.8}$$

A common denominator for (5.8) is

$$D(n,k) := P(n,k) \prod_{s\in[p]} (a_s n+b_s k+c_s)^{\underline{(a_s)^+ J+(b_s)^+ I}} \prod_{s\in[q]} (u_s n+v_s k+w_s+1)^{\overline{(-u_s)^+ J+(-v_s)^+ I}}$$

$$\times \prod_{s\in[p]} (a_s n + b_s k + c_s - b_s + 1)^{\overline{(b_s)^+}} \prod_{s\in[q]} (u_s n + v_s k + w_s - v_s)^{\underline{(-v_s)^+}}.$$

Thus, (5.8) is equivalent to

$$\frac{L_0 + L_1 + \cdots + L_J - R_1 + R_2}{D(n,k)} = 0, \tag{5.9}$$

where L_i's and R_i's are defined following (5.6) in §5.5.

Expanding (5.9), collecting the coefficients of like powers of k, and setting them to zero, we get a system of homogeneous linear equations with unknown polynomials in n, namely $a_0, a_1, \ldots, a_J, c_0, c_1, \ldots, c_{\mathcal{N}}$. Let us use $M\mathbf{x} = \mathbf{0}$ to represent the system. Let ν be $1 + \deg(\text{common numerator of } (5.9))$, then

$$\nu \le 1 + \deg_k P(n,k) + J(\mathcal{A} + (U - A)^+) + I(\mathcal{B} + (V - B)^+) + \mathcal{B}.$$

The matrix M is ν by $2 + J + \mathcal{N}$, of rank $\rho > 0$, and the ith row of M corresponds to the coefficient of k^{i-1} in the numerator of (5.9). Furthermore, $a_0(n)$ is assumed not to be identically zero. The stage is now set to apply the procedure for solving for $a_0(n)$ in §5.3.

In the solution set thus obtained, all of the a_j's and c_i's are either equal to 1 or are certain rational functions. To get a polynomial solution (that may have common polynomial-in-n factors), we multiply $\mathbf{x}$ by $\det M'$. Henceforth, we take $\det M_1'$ as a polynomial solution for $a_0(n)$.

Our goal is to bound real zeros of $a_0(n)$ from above for the following reasons. If $|a_0(n)| > 0$ for all $n \ge n_a$, then summing (5.7) over k yields a recurrence for $\sum_k F(n,k)$, i.e.,

$$a_0(n)\sum_k F(n,k) + a_1(n)\sum_k F(n-1,k) + \cdots + a_J(n)\sum_k F(n-J,k) = 0. \tag{5.10}$$

To show that 1 also satisfies the recurrence (5.10), i. e. that

$$a_0(n) + a_1(n) + \cdots + a_J(n) = 0 \qquad \forall n,$$

we use the fact that if a polynomial P of degree d has $d+1$ zeros, then $P = 0$. Therefore it suffices to show that 1 satisfies (5.10) for

$$n_0 \leq n \leq \max\{n_a + J - 1, n_0 + \max_{j \in [J]_0} \deg a_j(n)\}.$$

If it does, then we can use (5.10) to calculate 1 and $\sum_k F(n,k)$ in the following way:

$$1 = -\frac{a_1(n)f(n-1) + \cdots + a_J(n)f(n-J)}{a_0(n)},$$

and

$$\sum_k F(n,k) = -\frac{a_1(n)\sum_k F(n-1,k) + \cdots + a_J(n)\sum_k F(n-J,k)}{a_0(n)},$$

where $a_0(n) \neq 0$ $(n \geq n_a)$. We thus have

(a) $\sum_k F(n,k) = 1$ for $n_0 \leq n \leq \max\{n_a + J - 1, n_0 + \max_{j \in [J]_0} \deg a_j(n)\}$;

(b) both $\sum_k F(n,k)$ and 1 are uniquely defined for all $n \geq n_a$;

(c) both $\sum_k F(n,k)$ and 1 satisfy the same recurrence relation.

By induction, (a), (b), and (c) imply that $\sum_k F(n,k) = 1$ for all $n \geq n_0$.

We devote the rest of the proof to estimating $\deg \det M_1'$, $\max_{j \in [J]_0} \deg a_j(n)$, and n_1.

Observations.

(1) The entries of M are polynomials in n with integer coefficients because $P(n,k)$ is assumed to have integer coefficients, and w_s and c_s are fixed integer parameters.

(2) The maximum degrees of entries of M' and M_i' $(i \in [\rho])$ are bounded by the maximum degree of the entries of M because of the way we obtain M' and M_i' $(i \in [\rho])$ from M.

Let $a_{00} + a_{01}n + a_{02}n^2 + \cdots + a_{0d}n^d = a_0(n)$ (= $\det M_1'$). Then by Proposition 5.4, $a_0(n) \neq 0$, if

$$n > d \cdot \max_{0 \leq j \leq d-1} |a_{0j}|.$$

To find an upper bound for d, we use Lemma 5.5 which gives us a degree bound μ for the entries of M. From the second observation above, μ is also a degree bound for M_i' ($i \in [\rho]$) and M'. Thus

$$\deg \det M_i' \leq \rho\mu \leq \nu\mu \qquad i \in [\rho], \tag{5.11}$$

and

$$\deg \det M' \leq \rho\mu \leq \nu\mu,$$

where $\nu \leq 1 + \deg_k P(n,k) + J(\mathcal{A} + (U - A)^+) + I(\mathcal{B} + (V - B)^+) + \mathcal{B}$. From Step F of §5.3, we know that

$$\max_{j \in [J]_0} \deg a_j(n) \leq \max\{\deg \det M_1', \deg \det M_2', \ldots, \deg \det M_\rho', \deg \det M\} \leq \nu\mu.$$

To estimate $\max_{j \in [d]_0}\{|a_{0j}|\}$, we use Lemma 5.6 which gives us μ_8, a bound for the coefficients of the entries of M. From the way we obtained M_1' from M, the absolute value of the largest coefficient of M_1' is bounded by $(\mathcal{N} + J + 1)\mu_8$.

Finally we compute an upper bound for the absolute value of the largest coefficient of $\det M_1'$. Let $\omega := (\mathcal{N} + J + 1)\mu_8$ and $\mu := \max\{\mu_1, \mu_2\}$. In other words, ω is the computed upper bound for the absolute value of the largest coefficient of the entries of M_1'; and μ is the computed upper bound for the maximum degree of the entries of M_1'.

By the definition of the determinant, we have

$$\begin{aligned}\det M_1' &= \sum_{\sigma\in S_\rho} \operatorname{sgn}(\sigma) e_{1\sigma(1)} e_{2\sigma(2)} \cdots e_{\rho\sigma(\rho)} \\ &\preccurlyeq \sum_{\sigma\in S_\rho} |e_{1\sigma(1)} e_{2\sigma(2)} \cdots e_{\rho\sigma(\rho)}| \\ &\preccurlyeq \nu!\,\omega^\nu (n^\mu + n^{\mu-1} + \cdots + n + 1)^\nu.\end{aligned}$$

Hence

$$\max_i \left|[n^i] \det M_1'\right| \le \nu!\,\omega^\nu(\mu+1)^\nu. \tag{5.12}$$

Putting (5.11) and (5.12) together and using Proposition 5.4, we conclude that $a_0(n)$ does not vanish for all $n \ge \mu\nu \times \nu!\,\omega^\nu(\mu+1)^\nu (=: n_a)$. Knowing n_a, we calculate

$$n_1 = \max\{n_a + J - 1, n_0 + \max_{j\in[J]_0} \deg a_j(n)\}. \tag{5.13}$$

(See the discussion following formula (5.10) for the way we arrive at n_1.) Since

$$\max_{j\in[J]_0} \deg a_j(n) \le \nu\mu \ll n_a,$$

and n_0 is usually very small compared to ν, we can take n_1 to be $n_a + J - 1$ for Theorem 5.1. Otherwise, $n_1 = \max\{\mu\nu\nu!\,\omega^\nu(\mu+1)^\nu + J - 1, n_0 + \mu\nu\}$ will do. □

We next compute a cruder but simpler n_1. Given $F(n,k)$, an admissible proper-hypergeometric term, let $t_{lm} \in \mathbb{Z}$,

$$P(n,k) = \sum_{l=0}^{E}\sum_{m=0}^{D} t_{lm} k^l n^m, \quad \text{and} \quad F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{t=1}^{q}(u_t n + v_t k + w_t)!}\xi^k,$$

such that $F(n,k)$ satisfies a non-trivial recurrence relation, $\sum_{i,j}^{I,J} \alpha_{ij}(n) F(n-j, k-i) = 0$ for some positive integers I, J bounded by the result of Theorem 3.1 in [WZ3], or a sharper bound in Theorem 1.4. Let

$$x := \max\{|t_{lm}|, |a_s|, |b_s|, |c_s|, |u_t|, |v_t|, |w_t| : l \in [E]_0, m \in [D]_0, s \in [p], t \in [q]\}.$$

Then

$$\mu \leq x(p+q)(J+I+1)+D,$$

$$\mu_3 \leq (J+1)^D x,$$

$$\mu_4 \leq ((J+I+1)x)^{2(p+q)Ix},$$

$$\mu_5 \leq ((J+1)x)^{(p+q)Jx},$$

$$e_1 \leq (p+q)x(J+I+1),$$

$$e_2 \leq 2(p+q)x,$$

$$e_3 \leq (p+q)x,$$

$$\mu_6 \leq (x(J+I+1))^{2x(p+q)},$$

$$\mu_7 \leq (2x)^{x(p+q)},$$

$$\nu \leq 1+E+x(p+q)(J+I+1),$$

$$\mu_8 \leq (D+1)(E+1)(6x(J+I+1))^{2x(p+q)(I+J)+D+E},$$

$$\mathcal{N} \leq E+x(p+q)(J+I-1),$$

where the estimates are obtained directly from the expressions defining the variables in Section 5.5. Thus

$$\begin{aligned} \omega &:= (\mathcal{N}+J+1)\mu_8 \\ &\leq (E+x(p+q)(J+I-1)+J+1)(D+1)(E+1)(6x(J+I+1))^{2x(p+q)(I+J)+D+E}. \end{aligned}$$

Using the estimate obtained in the proof of Theorem 5.1, i.e., $n_1 := \mu\nu\cdot\nu!\,\omega^\nu(\mu+1)^\nu+J-1$, we get

$$n_1 = (f+h)(f+D)(f+h)!\,\left((f+g)(f+h+J)gh(6x(J+I+1))^{2f+D+E}\right)^{f+h},$$

where

$$f := x(p+q)(J+I+1),$$

$$g := D+1,$$

$$h := E+1.$$

Before proving the Main Theorem, we prove first the following corollary that is simpler than the Main Theorem, then we give the proof of the Main Theorem following the method used in the proof of Corollary 5.7.

Corollary 5.7. *Let*

$$F(n,k) = \frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k$$

be an admissible proper-hypergeometric term (free of $P(n,k)$), let

$$x := \max_s\{|a_s|,|b_s|,|c_s|,|u_s|,|v_s|,|w_s|\},$$

$$y := \max\{p,q\},$$

and let n_0 be a given integer. If $\sum_k F(n,k) = 1$ for

$$n_0 \le n \le (3xy)^{3(2xy)^6},$$

then $\sum_k F(n,k) = 1$ for all $n \ge n_0$.

Proof. From Theorem 1.4, we know that

$$J \le \mathcal{B} + (V-B)^+ \qquad \text{and}$$

$$I \le 1 + \delta + (\mathcal{A} + (U-A)^+ - 1)(\mathcal{B} + (V-B)^+),$$

where $\delta := \deg_k P(n,k)$,

$$U := \sum_{s:v_s \neq 0} u_s, \qquad V := \sum_s v_s, \qquad A := \sum_{s:b_s \neq 0} a_s, \qquad B := \sum_s b_s,$$
$$\mathcal{A} := \sum_{s:b_s \neq 0} (a_s)^+ + \sum_{s:v_s \neq 0} (-u_s)^+, \qquad \mathcal{B} := \sum_s (b_s)^+ + \sum_s (-v_s)^+.$$

Since $P(n,k) = 1$ in $F(n,k)$, we have that $\delta = 0$. Let

$$x := \max_s \{|a_s|, |b_s|, |c_s|, |u_s|, |v_s|, |w_s|\}$$

and $y := \max\{p, q\}$. Then

$$\begin{aligned} J &\leq \mathcal{B} + (V - B)^+ \\ &= \max\Big\{\sum_s (b_s)^+ + \sum_s (-v_s)^+, \quad \sum_s (-b_s)^+ + \sum_s (v_s)^+\Big\} \\ &\leq 2xy. \end{aligned}$$

Similarly,

$$I \leq 1 + (2xy - 1)(2xy) = (2xy)^2 - 2xy + 1 < (2xy)^2.$$

We express upper bounds for $\mathcal{N}$, e_1, e_2, e_3, μ, and μ_i, for all $i \in [8]$, in terms of x and y. (See §5.5 for the definitions of e_1, e_2, e_3, μ, and μ_i, for all $i \in [8]$.) From Step 3 of Chapter 2, we know that

$$\begin{aligned} \mathcal{N} &:= \deg_k P(n,k) + J(\mathcal{A} + (U - A)^+) + (I - 1)(\mathcal{B} + (V - B)^+) \\ &\leq 0 + (2xy)^2 + ((2xy)^2 - 2xy)(2xy) = (2xy)^3. \end{aligned}$$

Next we compute bounds for e_1, e_2, and e_3:

$$\begin{aligned} e_1 &:= J(\tilde{A} + (\tilde{U} - \tilde{A})^+) + (I + 1)\Big(\sum_s (b_s)^+ + \sum_s (-v_s)^+\Big) \\ &\leq (2xy)^2 + ((2xy)^2 - 2xy + 2)(2xy) = (2xy)^3 + 4xy, \end{aligned}$$

$$e_2 := 2\Big(\sum_s (b_s)^+ + \sum_s (-v_s)^+\Big) \le 4xy,$$
$$e_3 := \sum_s |b_s| + \sum_s |v_s| \le 2xy.$$

Now we compute the bounds for μ and μ_i, for $i \in [8]$:

$$\begin{aligned}
\mu_1 &= \deg_n P(n,k) + (I+1)\tilde{\mathfrak{B}} + J(\tilde{A} + (\tilde{U} - \tilde{A})^+) \\
&\le 0 + ((2xy)^2 - 2xy + 2)2xy + (2xy)(2xy) \\
&= (2xy)^3 + 4xy, \\
\mu_2 &= \max\Big\{ \sum_{s:a_s\neq 0} |b_s| + \sum_{s:u_s\neq 0} |v_s| \quad , \quad 2\sum_{s:a_s\neq 0} (b_s)^+ + 2\sum_{s:u_s\neq 0} (-v_s)^+ \Big\} \\
&\le 4xy, \\
\mu_3 &= 1 \qquad \text{because } P(n,k) = 1, \\
\mu_4 &\le (2x)^{2xy}((2xy+1)x)^{2xy((2xy)^2-2xy+1)}, \\
\mu_5 &\le ((2xy+1)x)^{2(2xy)xy}, \\
\mu_6 &\le ((J+I+1)x \cdot 2x)^{2xy} \\
&\le ((2xy + (2xy)^2 - 2xy + 2)2x^2)^{2xy} \\
&= \big(2((2xy)^2+2)x^2\big)^{2xy}, \\
\mu_7 &\le (2x)^{2xy}, \\
\mu_8 &:= \max\{3^{e_1}\mu_4\mu_5, 2^{\mathcal{N}}3^{e_3}\mu_7 + 3^{e_2}\mu_6\} \\
&\le \max\Big\{3^{(2xy)^3+4xy}(2x)^{2xy}((2xy+1)x)^{2xy((2xy)^2+1)} \quad , \\
&\qquad 2^{(2xy)^3}3^{2xy}(2x)^{2xy} + 3^{4xy}2^{2xy}(x^2((2xy)^2+2))^{2xy}\Big\} \\
&= 2^{2xy}3^{(2xy)^3+4xy}x^{2xy((2xy)^2+2)}(2xy+1)^{2xy((2xy)^2+1)},
\end{aligned}$$

$$\mu \leq \max\{\mu_1, \mu_2\}$$
$$\leq (2xy)^3 + 4xy.$$

From Theorem 5.1, we know that $n_1 = \max\{n_0 + \mu\nu, \mu\nu \cdot \nu! \omega^\nu (\mu+1)^\nu + J - 1\}$, where

$$\nu \leq 1 + \deg_k P(n,k) + J(\mathcal{A} + (U - A)^+) + I(\mathcal{B} + (V - B)^+) + \mathcal{B}$$

and

$$\omega = (\mathcal{N} + J + 1)\mu_8.$$

Estimating ν and ω in terms of x and y, we get

$$\nu \leq 1 + (2xy)^2 + ((2xy)^2 - 2xy + 1)2xy + 2xy$$
$$= 1 + 4xy + (2xy)^3,$$

and

$$\omega \leq ((2xy)^3 + 2xy + 1)2^{2xy}3^{(2xy)^3+4xy}x^{2xy((2xy)^2+2)}(2xy+1)^{2xy((2xy)^2+1)}.$$

Thus

$$n_1 \leq ((2xy)^3 + 4xy)((2xy)^3 + 4xy + 1)\left(((2xy)^3 + 4xy + 1)^2\omega\right)^\nu$$
$$< 3^{3(2xy)^6}x^{3(2xy)^6}y^{2(2xy)^6}$$
$$< (3xy)^{3(2xy)^6}. \quad \square$$

In practice n_0 and $\mu\nu$ are both much smaller than $(3xy)^{3(2xy)^6}$; therefore we take $(3xy)^{3(2xy)^6}$ as a bound for n_1.

We restate the

Main Theorem. *Let*

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k$$

be an admissible proper-hypergeometric term, and $P(n,k)$ be a polynomial with coefficients in $\mathbb{Z}$. Let

$$x := \max_s\{|a_s|, |b_s|, |c_s|, |u_s|, |v_s|, |w_s|\},$$

$$y := \max\{p, q\},$$

$$z := \max_{0\le i,j}\left|[n^j k^i]P(n,k)\right|,$$

$$d := 1 + \max\{\deg_k P(n,k), \deg_n P(n,k)\},$$

and let n_0 be a given integer. If $\sum_k F(n,k) = 1$ for

$$n_0 \le n \le (3xy)^{3(d+1)^2(2xy)^6} d^{5(d+1)(2xy)^3} z^{(d+1)(2xy)^3},$$

then $\sum_k F(n,k) = 1$ for all $n \ge n_0$.

Proof. From Theorem 1.4, we know that

$$J \le \mathcal{B} + (V - B)^+ \qquad \text{and}$$

$$I \le 1 + \delta + (\mathcal{A} + (U - A)^+ - 1)(\mathcal{B} + (V - B)^+),$$

where $\delta := \deg_k P(n,k)$,

$$U := \sum_{s:v_s\neq 0} u_s, \qquad V := \sum_s v_s, \qquad A := \sum_{s:b_s\neq 0} a_s, \qquad B := \sum_s b_s,$$

$$\mathcal{A} := \sum_{s:b_s\neq 0}(a_s)^+ + \sum_{s:v_s\neq 0}(-u_s)^+, \qquad \mathcal{B} := \sum_s (b_s)^+ + \sum_s (-v_s)^+.$$

We see that $J \le 2xy$, and that

$$I \le J + I \le d + (2xy)^2.$$

We express upper bounds for $\mathcal{N}$, e_1, e_2, e_3, μ, and μ_i for all $i \in [8]$, in terms of x, y, z, and d. (See §5.5 for the definitions of e_1, e_2, e_3, μ, and μ_i for all $i \in [8]$.) From Step 3 of Chapter 2 we know that

$$\begin{aligned}
\mathcal{N} &:= \deg_k P(n,k) + J(\mathcal{A} + (U - A)^+) + (I-1)(\mathcal{B} + (V - B)^+) \\
&< d + (2xy)(d - 1 + (2xy)^2) \\
&< (2xy)^3 + d(2xy + 1).
\end{aligned}$$

Next we compute bounds for e_1, e_2, and e_3:

$$\begin{aligned}
e_1 &\le J(\tilde{A} + (\tilde{U} - \tilde{A})^+) + (I+1)\Big(\sum_s (b_s)^+ + \sum_s (-v_s)^+\Big) \\
&\le (2xy)(d + 1 + (2xy)^2) = (2xy)^3 + (d+1)2xy, \\
e_2 &:= 2\Big(\sum_s (b_s)^+ + \sum_s (-v_s)^+\Big) \le 4xy, \\
e_3 &:= \sum_s |b_s| + \sum_s |v_s| \le 2xy.
\end{aligned}$$

Now we compute the bounds for μ_i, for $i \in [8]$:

$$\begin{aligned}
\mu_1 &= \deg_n P(n,k) + (I+1)\tilde{\mathcal{B}} + J(\tilde{A} + (\tilde{U} - \tilde{A})^+) \\
&\le d - 1 + (2xy)(d + 1 + (2xy)^2) \\
&= (2xy)^3 + (d+1)2xy + d - 1, \\
\mu_2 &= \max\Big\{ \sum_{s:a_s \ne 0} |b_s| + \sum_{s:u_s \ne 0} |v_s| \quad , \quad 2 \sum_{s:a_s \ne 0} (b_s)^+ + 2 \sum_{s:u_s \ne 0} (-v_s)^+ \Big\} \\
&\le 4xy,
\end{aligned}$$

$$\mu_3 = (1+J)^d z \le (1+2xy)^d z,$$

$$\mu_4 \le (2x)^{2xy}((2xy+1)x)^{2xy((2xy)^2-2xy+d)},$$

$$\mu_5 \le ((2xy+1)x)^{(2xy)^2},$$

$$\mu_6 \le ((J+I+1)2x^2)^{2xy} \le (2x^2)^{2xy}((2xy)^2+d+1)^{2xy},$$

$$\mu_7 \le (2x)^{2xy}.$$

Let $r := 2xy$. Then

$$\begin{aligned}
\mu_8 &\le \max\{d^2 3^{e_1}\mu_3\mu_4\mu_5, 2^{\mathcal{N}}3^{e_3}\mu_7 + 3^{e_2}\mu_6\} \\
&\le \max\Big\{d^2 3^{r^3+(d+1)r}(1+r)^d z(2x)^r((r+1)x)^{r(r^2+d)+r^2}, \\
&\qquad 2^{r^3+d(r+1)}3^r(2x)^r + 3^{2r}(2x^2)^r(r^2+d+1)^r \Big\} \\
&= \max\Big\{z2^r 3^{r^3+(d+1)r}d^2 x^{r^3+(d+1)r}(1+r)^{r^3+d(r+1)}, \\
&\qquad 2^{r^3+(d+1)r+d}(3x)^r + 2^r(3x)^{2r}(r^2+d+1)^r \Big\} \\
&= z2^r(3x)^{r^3+(d+1)r}d^2(1+r)^{r^3+d(r+1)}.
\end{aligned}$$

Thus

$$\mu \le \max\{\mu_1, \mu_2\} \le r^3 + (d+1)r + d + 1.$$

From Theorem 5.1, we know that $n_1 \le \mu\nu \cdot \nu!\,\omega^\nu(\mu+1)^\nu + J - 1$ where

$$\nu \le 1 + \deg_k P(n,k) + J(\mathcal{A} + (U-A)^+) + I(\mathcal{B} + (V-B)^+) + \mathcal{B}$$

and

$$\omega = (\mathcal{N} + J + 1)\mu_8.$$

Estimating ν and ω, we get

$$\nu \le d + r(d+r^2) + r = r^3 + r(d+1) + d,$$

where, still, $r = 2xy$, and

$$\omega \le (r^3 + d(r+1) + r + 1)z2^r(3x)^{r^3+(d+1)r}d^2(1+r)^{r^3+d(r+1)}.$$

Therefore

$$\begin{aligned} n_1 &< (r^3 + r(d+1) + d)^2(\nu^2\omega)^\nu \\ &< (r^3 + r(d+1) + d)^{2+2(r^3+r(d+1)+d)} \\ &\quad \times \left((r^3 + (d+1)(r+1))z2^r(3x)^{r^3+(d+1)r}d^2(1+r)^{r^3+d(r+1)}\right)^{r^3+r(d+1)+d} \\ &< ((d+1)r^3)^{3(d+1)r^3} z^{(d+1)r^3} 2^{(d+1)r^4} 3^{(d+1)^2r^6} d^{2(d+1)r^3} x^{(d+1)^2r^6} (1+r)^{(d+1)^2r^6} \\ &< (3x)^{3(d+1)^2r^6} d^{5(d+1)r^3} y^{2(d+1)^2r^6} z^{(d+1)r^3} \\ &< (3xy)^{3(d+1)^2r^6} d^{5(d+1)r^3} z^{(d+1)r^3}. \quad \square \end{aligned}$$

The following is a comparison of the 'sharp' and crude estimates in two relevant hypergeometric series.

Examples. First we calculate n_1 for $\sum_k \binom{n}{k} = 2^n$. Sharp $n_1 = 4 \times 3 \times 4!\,(4 \times 18)^4 + 1 < 10^{11}$, and crude $n_1 = 9 \times 10 \times 10!\,(50 \times 18^{12})^{10} < 10^{177}$.

Second we calculate n_1 for $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$. Sharp $n_1 = 12 \times 13 \times 13!\,(10 \times 13!/6!)^{13} < 10^{115}$, and crude $n_1 = 36 \times 37 \times 37!\,(37 \times 28 \times 36^{60})^{37} < 10^{3613}$.

5.7 Generalizations of Theorem 5.1

We consider in the following theorem hypergeometric identities of the type $\sum_k F(n,k) = f(n)$ where $F(n,k)$ is an admissible proper-hypergeometric term and $f(n)$ is a hypergeometric term. (Instead of $f = 1$ as in Theorem 5.1.) In Theorem 5.9, the object of interest will be identities of the form $\sum_k F(n,k) = \sum_k G(n,k)$ where F and G are both admissible proper-hypergeometric terms.

Theorem 5.8. *Let*

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k$$

be an admissible proper-hypergeometric term where P is a polynomial with coefficients in $\mathbb{Z}$. Let

$$f(n) = \frac{Q(n)}{S(n)}\frac{\prod_{s=1}^{r}(\alpha_s n + \beta_s)!}{\prod_{s=1}^{t}(\mu_s n + \nu_s)!}\zeta^n$$

be a hypergeometric term where Q and S are polynomials with coefficients in $\mathbb{Q}$. Let

$$x := \max_s\{|a_s|, |b_s|, |c_s|, |u_s|, |v_s|, |w_s|\},$$

$$y := \max\{p, q\},$$

$$z := \max_{0 \le i,j}\left|[n^j k^i]P(n,k)\right|,$$

$$d := 1 + \max\{\deg_k P(n,k), \deg_n P(n,k)\},$$

and let n_0 be a given integer. If $\sum_k F(n,k) = f(n)$ for $n_0 \le n \le n_1$, then $\sum_k F(n,k) = f(n)$ for <u>*all*</u> *$n \ge n_0$, where*

$$n_1 := \max\Big\{(3xy)^{3(d+1)^2(2xy)^6} d^{5(d+1)(2xy)^3} z^{(d+1)(2xy)^3},$$
$$n_0 + (d+1)^2(2xy)^6 + \deg Q + (2xy+1)\deg S + 2xy\Big(\sum_s |\alpha_s| + \sum_s |\mu_s|\Big)\Big\}.$$

Proof. By Theorem 3.1 of [WZ3], we know that there exist a positive integer $J \le \sum_s |b_s| + \sum_s |v_s|$ and polynomials $a_0(n)$, $a_1(n)$, ..., $a_J(n)$ such that

$$a_0(n)F(n,k) + a_1(n)F(n-1,k) + \cdots + a_J(n)F(n-J,k) = 0. \tag{5.14}$$

Since $F(n,k)$ is admissible, we can sum (5.14) over k to get

$$a_0(n)\sum_k F(n,k) + a_1(n)\sum_k F(n-1,k) + \cdots + a_J(n)\sum_k F(n-J,k) = 0.$$

From Theorem 5.1, we know that $a_0(n) \neq 0$ for all $n \geq n_a$. (See the line above (5.13) for the definition of n_a.) Therefore

$$\sum_k F(n,k) = -\frac{a_1(n)\sum_k F(n-1,k) + \cdots + a_J(n)\sum_k F(n-J,k)}{a_0(n)}$$

for all $n \geq n_a$. From the hypothesis, $\sum_k F(n,k) = f(n)$ for $n_0 \leq n \leq n_1$. Hence

$$f(n) = -\frac{a_1(n)f(n-1) + \cdots + a_J(n)f(n-J)}{a_0(n)} \tag{5.15}$$

for $n_a \leq n \leq n_1$. Dividing both sides of (5.15) by $f(n)$ we get

$$a_0(n) + a_1(n)\frac{f(n-1)}{f(n)} + a_2(n)\frac{f(n-2)}{f(n)} + \cdots + a_J(n)\frac{f(n-J)}{f(n)} = 0. \tag{5.16}$$

Putting (5.16) over a common denominator, we find that the numerator polynomial

$$a_0(n)f_0(n) + a_1(n)f_1(n) + \cdots + a_J(n)f_J(n) = 0 \tag{5.17}$$

where the f_j's ($j \in [J]_0$) are all polynomials of degree at most

$$\deg Q + (J+1)\deg S + J\Big(\sum_s |\alpha_s| + \sum_s |\mu_s|\Big)$$

and the a_j's ($j \in [J]_0$) are polynomials of degree at most $\nu\mu$. (See (5.11).) Since

$$n_1 \geq \max\Big\{n_a + J - 1 \quad , \quad n_0 + \nu\mu + \deg Q + \deg S + (J+1)\deg S + J\Big(\sum_s |\alpha_s| + \sum_s |\mu_s|\Big)\Big\},$$

we have more zeros than the degree of the polynomial in (5.17). Therefore, the numerator polynomial of (5.16) is identically zero, or equivalently,

$$a_0(n)f(n) + a_1(n)f(n-1) + \cdots + a_J(n)f(n-J) = 0.$$

Thus we have (5.15) for all $n \geq n_a$.

The facts that $f(n)$ and $\sum_k F(n,k)$ satisfy the same recurrence relation, and that $\sum_k F(n,k) = f(n)$ for $n_0 \le n \le n_1$ imply (by induction on n) that $\sum_k F(n,k) = f(n)$ for all $n \ge n_0$. □

The following theorem has a sum of hypergeometric terms on both sides of the equal sign.

Theorem 5.9. *Let*

$$F(n,k) = P(n,k)\frac{\prod_{s=1}^{p}(a_s n + b_s k + c_s)!}{\prod_{s=1}^{q}(u_s n + v_s k + w_s)!}\xi^k$$

and

$$G(n,k) = Q(n,k)\frac{\prod_{s=1}^{r}(\alpha_s n + \beta_s k + \gamma_s)!}{\prod_{s=1}^{t}(\delta_s n + \phi_s k + \psi_s)!}\zeta^k$$

be admissible proper-hypergeometric terms where P and Q are polynomials with coefficients in $\mathbb{Z}$. Let

$$x := \max_s\{|a_s|, |b_s|, |c_s|, |u_s|, |v_s|, |w_s|\}, \qquad f := \max_s\{|\alpha_s|, |\beta_s|, |\gamma_s|, |\delta_s|, |\phi_s|, |\psi_s|\},$$

$$y := \max\{p, q\}, \qquad g := \max\{r, t\},$$

$$z := \max_{0 \le i,j} \left|[n^j k^i]P(n,k)\right|,$$

$$d := 1 + \max\{\deg_k P(n,k), \deg_n P(n,k)\}, \qquad e := 1 + \max\{\deg_k Q(n,k), \deg_n Q(n,k)\},$$

let n_0 be a given integer, and assume wlog that $xy \le fg$. If $\sum_k F(n,k) = \sum_k G(n,k)$ for $n_0 \le n \le n_1$, then $\sum_k F(n,k) = \sum_k G(n,k)$ for <u>all</u> $n \ge n_0$, where

$$n_1 := \max\left\{n_0 + 2(\max\{d,e\} + 1)^2(2fg)^7 \quad , \quad (3xy)^{3(d+1)^2(2xy)^6} d^{5(d+1)(2xy)^3} z^{(d+1)(2xy)^3}\right\}$$

Proof. By Theorem 3.1 of [WZ3], there exist a positive integer $J \leq \sum_s |b_s| + \sum_s |v_s|$ and polynomials $a_0(n)$, $a_1(n)$, ... , $a_J(n)$ such that

$$a_0(n)F(n,k) + a_1(n)F(n-1,k) + \cdots + a_J(n)F(n-J,k) = 0. \tag{5.18}$$

Similarly, there exist a positive integer $I \leq \sum_s |\beta_s| + \sum_s |\phi_s|$ and polynomials $b_0(n)$, $b_1(n)$, ... , $b_I(n)$ such that

$$b_0(n)G(n,k) + b_1(n)G(n-1,k) + \cdots + b_I(n)G(n-I,k) = 0. \tag{5.19}$$

Summing (5.18) and (5.19) over k, we get

$$a_0(n)\sum_k F(n,k) + a_1(n)\sum_k F(n-1,k) + \cdots + a_J(n)\sum_k F(n-J,k) = 0 \tag{5.20}$$

and

$$b_0(n)\sum_k G(n,k) + b_1(n)\sum_k G(n-1,k) + \cdots + b_I(n)\sum_k G(n-I,k) = 0, \tag{5.21}$$

since both F and G are admissible.

By the hypothesis, $xy \leq fg$, so both I and J are bounded by $2fg$. Our goal is to show that $\sum_k G(n,k)$ satisfies the same recurrence relation as $\sum_k F(n,k)$, or visa versa. This is achieved if $\sum_k G(n,k)$ satisfies (5.20) or $\sum_k F(n,k)$ satisfies (5.21) for

$$n_0 \leq n \leq n_0 + (2fg+1)\left(1 + \max\left\{\max_{j\in[J]_0} \deg a_j(n), \max_{i\in[I]_0} \deg b_i(n)\right\}\right),$$

because there are $(I+1)(1 + \max_{i\in[I]_0} \deg b_i(n))$ indeterminates in

$$\sum_{\substack{i\in[I]_0 \\ 0\leq r\leq \max_{j\in[I]_0} \deg b_j(n)}} c_{r,i} n^r \sum_k G(n-i,k) = 0,$$

and $(J+1)(1+\max_{j\in[J]_0} \deg a_j(n))$ indeterminates in the corresponding recurrence for $F(n,k)$.

We know from Theorem 5.1 that

$$I \le 2fg \qquad \text{and} \qquad \max_{i\in[I]_0} \deg b_i(n) \le (e+1)^2(2fg)^6;$$

$$J \le 2xy \le 2fg \qquad \text{and} \qquad \max_{j\in[J]_0} \deg a_j(n) \le (d+1)^2(2xy)^6 \le (d+1)^2(2fg)^6.$$

Thus

$$\max\left\{(I+1)(1+\max_{i\in[I]_0} \deg b_i(n)) \quad , \quad (J+1)(1+\max_{j\in[J]_0} \deg a_j(n))\right\} \le n_2,$$

where

$$n_2 := (2fg+1)\left(1+\max\left\{\max_{j\in[J]_0} \deg a_j(n) \quad , \quad \max_{i\in[I]_0} \deg b_i(n)\right\}\right).$$

Since $n_1 - n_0 > n_2$, and $\sum_k F(n,k) = \sum_k G(n,k)$ for $n_0 \le n \le n_1$, we conclude that $\sum_k G(n,k)$ satisfies the same recurrence relation as $\sum_k F(n,k)$. Further, $a_0(n) \ne 0$ for $n \ge n_a$. Therefore, both $\sum_k F(n,k)$ and $\sum_k G(n,k)$ are determined inductively by

$$\sum_k F(n,k) = -\frac{a_1(n)\sum_k F(n-1,k) + \cdots + a_J(n)\sum_k F(n-J,k)}{a_0(n)}$$

for $n \ge n_a$. We conclude that if $\sum_k F(n,k) = \sum_k G(n,k)$ for $n_0 \le n \le n_1$, then $\sum_k F(n,k) = \sum_k G(n,k)$ for all $n \ge n_0$. $\square$

CHAPTER VI

MULTIVARIABLE HYPERGEOMETRIC IDENTITIES ARE ALMOST TRIVIAL

In this chapter, we generalize the result of Chapter 5 to r variables. The lemmas and the proof of the following theorem parallel those in Chapter 5 closely.

Theorem 6.1. *Let*

$$F(n,\mathbf{k}) = P(n,\mathbf{k})\frac{\prod_{s=1}^{p}(a_s n + \mathbf{b}_s \cdot \mathbf{k} + c_s)!}{\prod_{s=1}^{q}(u_s n + \mathbf{v}_s \cdot \mathbf{k} + w_s)!}\mathbf{z}^{\mathbf{k}} \tag{6.1}$$

be an admissible proper-hypergeometric term, and let $P(n,\mathbf{k})$ be a polynomial with coefficients in $\mathbb{Q}$. Then given n_0, there exists an effectively computable positive integer n_1 such that if $\sum_{\mathbf{k}} F(n,\mathbf{k}) = 1$ for all $n_0 \le n < n_1$, then $\sum_{\mathbf{k}} F(n,\mathbf{k}) = 1$ for all $n \ge n_0$.

With the same observation as the one made after Theorem 5.1, it suffices to prove Theorem 6.1 for those polynomials $P(n,\mathbf{k})$ with integer coefficients.

6.1 Two approximation lemmas

Notation. We use $[n]$ to denote $\{1,2,\ldots,n\}$, $[n]_0$ to denote $\{0\} \cup [n]$, and $\mathbf{l} \in [\mathbf{E}]_0$ to denote $l_1 \in [E_1]_0, \ldots, l_r \in [E_r]_0$. All the bold faced letters stand for an r dimensional vector. We use $[x^n\mathbf{y}^{\mathbf{l}}]P(x,\mathbf{y})$ to denote the coefficient of $x^n\mathbf{y}^{\mathbf{l}}$ in $P(x,\mathbf{y})$. As in Chapter 5, $P(n,\mathbf{k}) \preccurlyeq Q(n,\mathbf{k})$ means that for all $(m,\mathbf{l})$, $|[n^m\mathbf{k}^{\mathbf{l}}]P(n,\mathbf{k})| \le |[n^m\mathbf{k}^{\mathbf{l}}]Q(n,\mathbf{k})|$.

We need the following lemmas for the proof of Theorem 6.1.

Lemma 6.2. *Let $P(n,\mathbf{k})$ be a polynomial in n and $\mathbf{k}$ with integer coefficients. And let*

$$\mu = \max_{\mathbf{l}\in[\mathbf{E}]_0, m\in[D]_0} \left|[n^m\mathbf{k}^\mathbf{l}]P(n,\mathbf{k})\right|, \quad D = \deg_n P(n,\mathbf{k}), \quad \textit{and} \quad E_i = \deg_{k_i} P(n,\mathbf{k}), i \in [r].$$

Then for every positive integer J,

$$\max_{\mathbf{l}\in[\mathbf{E}]_0, m\in[D]_0, j\in[J]_0} \left|[n^m\mathbf{k}^\mathbf{l}]P(n-j,\mathbf{k})\right| \le (1+J)^D\mu.$$

Proof. Suppose $P(n,\mathbf{k}) = \sum_{\mathbf{l}=0}^{\mathbf{E}} \sum_{m=0}^{D} t_{\mathbf{l}m}\mathbf{k}^\mathbf{l} n^m$. For some fixed $\mathbf{l}$ and m, we have that $|[\mathbf{k}^\mathbf{l} n^m]P(n-j,\mathbf{k})|$ is

$$\begin{aligned}\left|\sum_{i=0}^{D-m}(-1)^i t_{\mathbf{l},m+i}\binom{m+i}{m} j^i\right| &\le \sum_{i=0}^{D}\binom{D}{i} J^i \max_{\mathbf{l}\in[\mathbf{E}]_0, m\in[D]_0}\left|[n^m\mathbf{k}^\mathbf{l}]P(n,\mathbf{k})\right| \\ &= (1+J)^D\mu\end{aligned}$$

for all $j \in [J]_0$, $\mathbf{l} \in [\mathbf{E}]_0$ and $m \in [D]_0$. □

Lemma 6.3. *Let $Q(n,\mathbf{k}) = \prod_{s=1}^{q}(a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s)$, where a_s and c_s are integers, and $\mathbf{b}_s \in \mathbb{Z}^r (s \in [q])$. Then*

$$\max_{m\in[q]_0, \mathbf{l}\in[\mathbf{q}]_0} \left|[n^m\mathbf{k}^\mathbf{l}]Q(n,\mathbf{k})\right| < (r+2)^q \prod_{s\in[q]} \max\{|a_s|, |b_{1s}|, |b_{2s}|, \ldots, |b_{rs}|, |c_s|\}.$$

Proof. We know that

$$\begin{aligned} Q(n,\mathbf{k}) &= \prod_{s=1}^{q}(a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s) \\ &\preccurlyeq (n + \mathbf{k}\cdot\mathbf{1} + 1)^q \prod_{s\in[q]} \max\{|a_s|, |b_{1s}|, |b_{2s}|, \ldots, |b_{rs}|, |c_s|\}. \end{aligned}$$

Since the absolute value of the largest coefficient of $(n + \mathbf{k}\cdot\mathbf{1} + 1)^q$ is less than $(r+2)^q$,

$$\max_{m\in[q]_0, \mathbf{l}\in[\mathbf{q}]_0} \left|[n^m\mathbf{k}^\mathbf{l}]Q(n,\mathbf{k})\right| < (r+2)^q \prod_{s\in[q]} \max\{|a_s|, |b_{1s}|, |b_{2s}|, \ldots, |b_{rs}|, |c_s|\}. \quad \square$$

6.2 The leading coefficient, $a_0(n)$, of the recurrence

In this section, we estimate the degree and the largest coefficient of the leading coefficient, $a_0(n)$, a polynomial in n, in the recurrence of $F(n,\mathbf{k})$. With the upper bounds for the degree and the largest coefficient, we compute an upper bound for the positive integer with the property that for all $n \geq n_1$, $a_0(n) \neq 0$. Thus the proof of Theorem 6.1 is complete.

The plan for achieving this goal parallels that of §5.5, and consists of the following four stages:

Stage 1. Take a given admissible proper-hypergeometric term $F(n,\mathbf{k})$, and use Theorem 4.2A of [WZ3] to say that $F(n,\mathbf{k})$ satisfies a recurrence of the form:

$$a_0(n)F(n,\mathbf{k}) + a_1(n)F(n-1,\mathbf{k}) + \cdots + a_J(n)F(n-J,\mathbf{k}) = \sum_{i=1}^{r} \Delta_i G_i(n,\mathbf{k}), \tag{6.2}$$

where the $a_j(n)$'s are unknown polynomials in n. Divide (6.2) by $\hat{F}(n,\mathbf{k})$ ($:= \frac{F(n,\mathbf{k})}{P(n,\mathbf{k})}$) and put the resulting sum of rational functions over a common denominator.

Stage 2. Equate to 0 the coefficient of each monomial of $\mathbf{k}$ in the common numerator, and solve the resulting homogeneous linear equations for the unknowns $a_j(n)$'s and $c_i(\mathbf{e},n)$'s (see (6.3) below for $c_i(\mathbf{e},n)$'s) by Cramer's rule for $a_0(n)$ only, in the form

$$a_0(n) = \frac{\det M_1'}{\det M'}.$$

(See §5.3 for the way M' and M_1' are obtained.)

Stage 3. Observe that $a_0(n) = 0$ exactly when $\det M_1' = 0$. Therefore, we express $\det M_1'$ as a polynomial in n, and obtain an upper bound for the degree of det M_1' (see §5.6 formula (5.11)) and the largest coefficient of det M_1' (see §5.6 formula (5.12)).

Stage 4. Use the simple fact that if $f(x)$ is a polynomial over $\mathbb{Z}$, d is the degree of $f(x)$ and m is $\max_{i\in[d]_0} |[x^i]f(x)|$, then $f(x) \neq 0$ for all $x > md$. (See Proposition 5.4 in §5.4.) Thus we use the estimates in Stage 3 to obtain an n_a such that for all $n > n_a$, $a_0(n) \neq 0$.

We now proceed to do Stage 1 of the plan in detail. Let an admissible proper-hypergeometric term $F(n, \mathbf{k})$ be given such that $P(n, \mathbf{k})$ in $F(n, \mathbf{k})$ has integer coefficients. (See formula (6.1) for the definition of $F(n, \mathbf{k})$.) Then Theorem 4.2A of [WZ3] guarantees us the existence of polynomials $a_0(n)$, $a_1(n), \ldots, a_J(n)$, not all zero,

$$J \leq \left\lfloor \frac{1}{r!}\Big(\sum_s \sum_{r'} |(\mathbf{b}_s)_{r'}| + \sum_s \sum_{r'} |(\mathbf{v}_s)_{r'}|\Big)^r \right\rfloor,$$

and hypergeometric functions $G_1(n, \mathbf{k}), \ldots, G_r(n, \mathbf{k})$ such that $G_i(n, \mathbf{k}) = R_i(n, \mathbf{k})F(n, \mathbf{k})$ for rational functions R_i ($i \in [r]$) and such that

$$a_0(n)F(n, \mathbf{k}) + a_1(n)F(n-1, \mathbf{k}) + \cdots + a_J(n)F(n-J, \mathbf{k}) = \sum_{i=1}^{r} \Delta_i G_i(n, \mathbf{k}), \tag{6.2}$$

where $\Delta_i G_i(n, \mathbf{k}) = G_i(n, \mathbf{k}) - G_i(n, k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$.

Without loss of generality, assume $a_0(n)$ is not identically zero. From Chapter 4, we know that $R_i(n, \mathbf{k})$ is of the form

$$\sum_{\substack{0 \leq \mathbf{e} \leq (\mathcal{N}_i, \mathcal{N}_i, \ldots, \mathcal{N}_i) \\ \mathbf{e}\cdot\mathbf{1} \leq \mathcal{N}_i}} \frac{c_i(\mathbf{e}, n)\mathbf{k}^{\mathbf{e}}}{D_{R_i}(n, \mathbf{k})} \tag{6.3}$$

for some unknown polynomials in n, namely $c_i(\mathbf{e}, n)$, where

$$\mathcal{N}_i := \deg_{\mathbf{k}} P(n, \mathbf{k}) + (I_i - 1)(\mathcal{B}_i + (V_i - B_i)^+) + J(\mathcal{A} + (U - A)^+) + \sum_{i < t \leq r} I_t(\mathcal{B}_t + (V_t - B_t)^+),$$

and

$$D_{R_i}(n,\mathbf{k}) = P(n,\mathbf{k}) \prod_{s=1}^{p} (a_s n + \mathbf{b}_s \cdot \mathbf{k} + c_s)^{\underline{(a_s)^+ J + (I_i - 1)(b_{is})^+ + \sum_{i<t\le r} I_t (b_{ts})^+}}$$
$$\times \prod_{s=1}^{q} (u_s n + \mathbf{v}_s \cdot \mathbf{k} + w_s + 1)^{\overline{(-u_s)^+ J + (I_i - 1)(-v_{is})^+ + \sum_{i<t\le r} I_t(-v_{ts})^+}}.$$

(See Chapter 3 for the definitions of $\mathcal{A}, \mathcal{B}_i, A, B_i, U$ and V_i ($i \in [r]$).)

Let $\mathbf{k}_i$ denote $(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$. To eliminate the factorials in (6.2), we divide both sides of (6.2) by $\hat{F}(n,\mathbf{k}) := F(n,\mathbf{k})/P(n,\mathbf{k})$ to get

$$\frac{\sum_{j=0}^{J} a_j(n) F(n-j,\mathbf{k})}{\hat{F}(n,\mathbf{k})} = \sum_{i\in[r]} \left(R_i(n,\mathbf{k}) P(n,\mathbf{k}) - R_i(n,\mathbf{k}_i) \frac{F(n,\mathbf{k}_i)}{\hat{F}(n,\mathbf{k})} \right)$$
$$= \sum_{i\in[r]} \left(\frac{\sum_{\mathbf{e}} c_i(\mathbf{e},n)\mathbf{k}^{\mathbf{e}}}{\hat{D}_{R_i}(n,\mathbf{k})} - \frac{\sum_{\mathbf{e}} c_i(\mathbf{e},n)\mathbf{k}_i^{\mathbf{e}} P(n,\mathbf{k}_i)\hat{F}(n,\mathbf{k}_i)}{\hat{D}_{R_i}(n,\mathbf{k}_i) P(n,\mathbf{k}_i) \hat{F}(n,\mathbf{k})} \right),$$

where $\hat{D}_{R_i}(n,\mathbf{k}) := D_{R_i}(n,\mathbf{k})/P(n,\mathbf{k})$. In order to avoid writing the hat, we use $D_{R_i}(n,\mathbf{k})$ to denote $\hat{D}_{R_i}(n,\mathbf{k})$, and with this notation, we use $R_i(n,\mathbf{k})$ to mean

$$\sum_{\substack{0 \le \mathbf{e} \le (\mathcal{N}_i, \mathcal{N}_i, \ldots, \mathcal{N}_i) \\ \mathbf{e}\cdot\mathbf{1} \le \mathcal{N}_i}} \frac{c_i(\mathbf{e},n)\mathbf{k}^{\mathbf{e}}}{D_{R_i}(n,\mathbf{k})}.$$

We find a common denominator of

$$(6.4) \quad a_0(n)P(n,\mathbf{k}) + a_1(n)\frac{F(n-1,\mathbf{k})}{\hat{F}(n,\mathbf{k})} + \cdots + a_J(n)\frac{F(n-J,\mathbf{k})}{\hat{F}(n,\mathbf{k})}$$
$$= \sum_i \left(\sum_{\substack{0 \le \mathbf{e} \le (\mathcal{N}_i, \mathcal{N}_i, \ldots, \mathcal{N}_i) \\ \mathbf{e}\cdot\mathbf{1} \le \mathcal{N}_i}} \frac{c_i(\mathbf{e},n)\mathbf{k}^{\mathbf{e}}}{D_{R_i}(n,\mathbf{k})} \right.$$
$$- \sum_{\substack{0 \le \mathbf{e} \le (\mathcal{N}_i, \mathcal{N}_i, \ldots, \mathcal{N}_i) \\ \mathbf{e}\cdot\mathbf{1} \le \mathcal{N}_i}} \left(\frac{c_i(\mathbf{e},n) k_1^{e_1} \cdots (k_i - 1)^{e_i} \cdots k_r^{e_r}}{D_{R_i}(n, k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)} \right.$$
$$\left.\left. \times \frac{\hat{F}(n, k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)}{\hat{F}(n,\mathbf{k})} \right) \right).$$

The computation involved for finding a common denominator of (6.4) consists of finding a common denominator for LHS of (6.4), each summand of RHS of (6.4), all of RHS of (6.4) and finally finding the least common multiple of the denominator.

First we note that a common denominator of the LHS for (6.4) is

$$D_{\text{LHS}} := \prod_{s=1}^{p}(a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s)^{\underline{(a_s)^+J}} \times \prod_{s=1}^{q}(u_s n + \mathbf{v}_s\cdot\mathbf{k} + w_s + 1)^{\overline{(-u_s)^+J}},$$

and D_{LHS} divides D_{R_i} for all $i \in [r]$. Therefore, it suffices to find a common denominator of RHS of (6.4). To do so, we first find a common denominator for every term of the summand of RHS of (6.4), namely,

$$R_i(n,\mathbf{k}) - R_i(n,\mathbf{k}_i)\frac{\hat{F}(n,\mathbf{k}_i)}{\hat{F}(n,\mathbf{k})}. \tag{6.5}$$

Since

$$D_{R_i}(n,\mathbf{k}) = \prod_{s=1}^{p}(a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s)^{\underline{(a_s)^+J+(I_i-1)(b_{is})^+ + \sum_{i<t\le r} I_t(b_{ts})^+}}$$
$$\times \prod_{s=1}^{q}(u_s n + \mathbf{v}_s\cdot\mathbf{k} + w_s + 1)^{\overline{(-u_s)^+J+(I_i-1)(-v_{is})^+ + \sum_{i<t\le r} I_t(-v_{ts})^+}},$$

we replace k_i by $k_i - 1$ to get

$$D_{R_i}(n,\mathbf{k}_i) = \prod_{s=1}^{p}(a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s - b_{is})^{\underline{(a_s)^+J+(I_i-1)(b_{is})^+ + \sum_{i<t\le r} I_t(b_{ts})^+}}$$
$$\times \prod_{s=1}^{q}(u_s n + \mathbf{v}_s\cdot\mathbf{k} + w_s + 1 - v_{is})^{\overline{(-u_s)^+J+(I_i-1)(-v_{is})^+ + \sum_{i<t\le r} I_t(-v_{ts})^+}}.$$

Furthermore, a denominator for $\frac{\hat{F}(n,\mathbf{k}_i)}{\hat{F}(n,\mathbf{k})}$ is

$$\prod_{s=1}^{p}(a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s)^{\underline{(b_{is})^+}} \times \prod_{s=1}^{q}(u_s n + \mathbf{v}_s\cdot\mathbf{k} + w_s + 1)^{\overline{(-v_{is})^+}}.$$

Therefore, a common denominator for (6.5) is

$$(6.6)\quad \prod_{s=1}^{p}(a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s)^{\underline{(a_s)^+ J + \sum_{i\le t\le r} I_t(b_{ts})^+}}$$
$$\times \prod_{s=1}^{q}(u_s n + \mathbf{v}_s\cdot\mathbf{k} + w_s + 1)^{\overline{(-u_s)^+ J + \sum_{i\le t\le r} I_t(-v_{ts})^+}}$$
$$\times \prod_{s=1}^{p}(a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s)^{\underline{(b_{is})^+}} \times \prod_{s=1}^{q}(u_s n + \mathbf{v}_s\cdot\mathbf{k} + w_s + 1)^{\overline{(-v_{is})^+}}.$$

Let $A_s = a_s n + \mathbf{b}_s\cdot\mathbf{k} + c_s$ and $U_s = u_s n + \mathbf{v}_s\cdot\mathbf{k} + w_s$. Putting (6.5) and (6.6) together, we conclude that a common denominator for the RHS of (6.4) is

$$D(n,\mathbf{k}) := \prod_{s=1}^{p}(A_s)^{\underline{(a_s)^+ J + \sum_{1\le t\le r} I_t(b_{ts})^+}} \times \prod_{s=1}^{q}(U_s+1)^{\overline{(-u_s)^+ J + \sum_{1\le t\le r} I_t(-v_{ts})^+}}$$
$$\times \prod_{s=1}^{p}(A_s)^{\underline{\max_{i\in[r]}(b_{is})^+}} \times \prod_{s=1}^{q}(U_s+1)^{\overline{\max_{i\in[r]}(-v_{is})^+}}.$$

Putting (6.4) over $D(n,\mathbf{k})$, and collecting all terms to the LHS, we get

$$(6.7)\qquad \frac{L_0 + L_1 + \cdots + L_J - \sum_{i\in[r]}(R_{i1} - R_{i2})}{D(n,\mathbf{k})} = 0,$$

where

$$\begin{aligned}
L_j &= \frac{a_j(n)F(n-j,\mathbf{k})}{\hat{F}(n,\mathbf{k})} D(n,\mathbf{k}) \\
&= \frac{a_j(n)P(n-j,\mathbf{k})\prod_{s=1}^{p}(A_s+1)^{\overline{j(-a_s)^+}}\prod_{s=1}^{q}(U_s)^{\underline{j(u_s)^+}}}{\prod_{s=1}^{p}(A_s)^{\underline{j(a_s)^+}}\prod_{s=1}^{q}(U_s+1)^{\overline{j(-u_s)^+}}} \\
&\quad \times \prod_{s=1}^{p}(A_s)^{\underline{\max_{i\in[r]}(b_{is})^+}} \prod_{s=1}^{q}(U_s+1)^{\overline{\max_{i\in[r]}(-v_{is})^+}} \\
&\quad \times \prod_{s=1}^{p}(A_s)^{\underline{(a_s)^+ J + \sum_{1\le t\le r} I_t(b_{ts})^+}} \prod_{s=1}^{q}(U_s+1)^{\overline{(-u_s)^+ J + \sum_{1\le t\le r} I_t(-v_{ts})^+}} \\
&= a_j(n)P(n-j,\mathbf{k})\prod_{s=1}^{p}(A_s+1)^{\overline{j(-a_s)^+}}\prod_{s=1}^{q}(U_s)^{\underline{j(u_s)^+}}\prod_{s=1}^{p}(A_s)^{\underline{\max_{i\in[r]}(b_{is})^+}} \\
&\quad \times \prod_{s=1}^{q}(U_s+1)^{\overline{\max_{i\in[r]}(-v_{is})^+}}\prod_{s=1}^{p}(A_s - j(a_s)^+)^{\underline{(a_s)^+(J-j) + \sum_{1\le t\le r} I_t(b_{ts})^+}} \\
&\quad \times \prod_{s=1}^{q}(U_s + j(-u_s)^+ + 1)^{\overline{(-u_s)^+(J-j) + \sum_{1\le t\le r} I_t(-v_{ts})^+}},
\end{aligned}$$

for $j \in [J]_0$, and for $i \in [r]$,

$$R_{i1} = D(n,\mathbf{k}) \times \sum_{\substack{\mathbf{0}\le\mathbf{e}\le(\mathcal{N}_i,\mathcal{N}_i,\ldots,\mathcal{N}_i)\\ \mathbf{e}\cdot\mathbf{1}\le\mathcal{N}_i}} \frac{c_i(\mathbf{e},n)\mathbf{k}^{\mathbf{e}}}{D_{R_i}(n,\mathbf{k})}$$

$$= \left(\sum_{\substack{\mathbf{0}\le\mathbf{e}\le(\mathcal{N}_i,\mathcal{N}_i,\ldots,\mathcal{N}_i)\\ \mathbf{e}\cdot\mathbf{1}\le\mathcal{N}_i}} c_i(\mathbf{e},n)\mathbf{k}^{\mathbf{e}}\right) \prod_{s=1}^{p}(A_s)^{\overline{\max_{i\in[r]}(b_{is})^+}} \prod_{s=1}^{q}(U_s+1)^{\overline{\max_{i\in[r]}(-v_{is})^+}}$$

$$\times \prod_{s=1}^{p}\left(A_s - J(a_s)^+ - \sum_{1\le t\le r} I_t(b_{ts})^+ + 1\right)^{\overline{(b_{is})^+ + \sum_{1\le t<i} I_t(b_{ts})^+}}$$

$$\times \prod_{s=1}^{q}\left(U_s + J(-u_s)^+ + \sum_{1\le t\le r} I_t(-v_{ts})^+\right)^{\overline{(-v_{is})^+ + \sum_{1\le t<i} I_t(-v_{ts})^+}},$$

and

$$R_{i2} = D(n,\mathbf{k})\frac{\hat{F}(n,\mathbf{k}_i)}{\hat{F}(n,\mathbf{k})} \sum_{\substack{\mathbf{0}\le\mathbf{e}\le(\mathcal{N}_i,\mathcal{N}_i,\ldots,\mathcal{N}_i)\\ \mathbf{e}\cdot\mathbf{1}\le\mathcal{N}_i}} \frac{c_i(\mathbf{e},n)\mathbf{k}_i^{\mathbf{e}}}{D_{R_i}(n,\mathbf{k}_i)}$$

$$= \left(\sum_{\substack{\mathbf{0}\le\mathbf{e}\le(\mathcal{N}_i,\mathcal{N}_i,\ldots,\mathcal{N}_i)\\ \mathbf{e}\cdot\mathbf{1}\le\mathcal{N}_i}} \left(\prod_{l:l\ne i} k_l^{e_l} \sum_{j\le e_i}(-1)^j c_i(\mathbf{e},n)\binom{e_i}{j}k_i^{e_i-j}\right)\right)$$

$$\times \prod_{s=1}^{p}(A_s+1)^{\overline{(-b_{is})^+}} \prod_{s=1}^{q} U_s^{\overline{v_{is}^+}} \prod_{s=1}^{p}(A_s)^{\overline{b_{is}^+}} \prod_{s=1}^{q}(U_s+1)^{\overline{(-v_{is})^+}}$$

$$\times \prod_{s=1}^{p}\left(A_s - \max_{i\in[r]}(b_{is})^+ + 1\right)^{\overline{-b_{is}^+ + \max_{i\in[r]}(b_{is})^+}}$$

$$\times \prod_{s=1}^{q}\left(U_s + \max_{i\in[r]}(-v_{is})^+\right)^{\overline{-(-v_{is})^+ + \max_{i\in[r]}(-v_{is})^+}}$$

$$\times \prod_{s=1}^{p}\left(A_s - J(a_s)^+ - \sum_{1\le t\le r} I_t(b_{ts})^+\right)^{\overline{\sum_{1\le t<i} I_t(b_{ts})^+}}$$

$$\times \prod_{s=1}^{q}\left(U_s + J(-u_s)^+ + \sum_{1\le t\le r} I_t(-v_{ts})^+ + 1\right)^{\overline{\sum_{1\le t<i} I_t(-v_{ts})^+}}.$$

In Stage 2, we solve for the unknown polynomials, $a_0(n),\ldots,a_J(n)$ and $c_i(\mathbf{e},n)$ ($i \in [r]$, and $\mathbf{0} \le \mathbf{e} \le (\mathcal{N}_i,\mathcal{N}_i,\ldots,\mathcal{N}_i)$ for $\mathbf{e}\cdot\mathbf{1} \le \mathcal{N}_i$), we expand the terms of (6.7), and collect like monomials of $\mathbf{k}$. Since the LHS of (6.7) is zero, the coefficients of all $\mathbf{k}^{\mathbf{e}}$ must be identically zero. This yields a system of linear homogeneous equations. We can express the system in matrix form as $M\mathbf{x} = \mathbf{0}$ for $\mathbf{x}^t = (a_0, a_1, \ldots, a_J, c_i(\mathbf{e}, n))$. To solve for $a_0(n)$, we apply the procedure in §5.3, and get

$$a_0(n) = \frac{\det M_1'}{\det M'}.$$

We are now ready to do Stage 3 of the plan. First, we find an upper bound for the maximum degree of the entries of M. Second, we find an upper bound for the largest coefficient of the entries of M. Third we find the size of M. Finally, we find upper bounds for $\deg\det M_1'$ and $\max_i |[n^i]\det M_1'|$. We use $\tilde{\mathfrak{B}}$ to mean $(\tilde{\mathfrak{B}}_1, \tilde{\mathfrak{B}}_2, \ldots, \tilde{\mathfrak{B}}_r)$; and $(\mathbf{b}_s)^+ = (b_{1s}^+, b_{2s}^+, \ldots, b_{rs}^+)$.

Step 1. An upper bound for the maximum degree over all the entries of M regarded as polynomials in n.

Lemma 6.4. *Let*

$$\mu_1 := \deg_n P(n, \mathbf{k}) + \mathbf{I} \cdot \tilde{\mathfrak{B}} + J(\tilde{A} + (\tilde{U} - \tilde{A})^+) + \sum_{s: a_s \neq 0} \max_l b_{ls}^+ + \sum_{s: u_s \neq 0} \max_l (-v_{ls})^+,$$

where for $l \in [r]$,

$$\tilde{\mathfrak{B}}_l = \sum_{s: a_s \neq 0} b_{ls}^+ + \sum_{s: u_s \neq 0} (-v_{ls})^+, \quad \tilde{\mathcal{A}} = \sum_{s \in [p]} a_s^+ + \sum_{s \in [q]} (-u_s)^+,$$
$$\tilde{U} = \sum_{s \in [q]} u_s, \qquad \tilde{A} = \sum_{s \in [p]} a_s,$$

and let

$$\mu_2 := \sum_{s: a_s \neq 0} \max_l b_{ls}^+ + \sum_{s: u_s \neq 0} \max_l (-v_{ls})^+ + \max_{l \in [r]} \left\{ \tilde{\mathfrak{B}}_l + (\tilde{V}_l - \tilde{B}_l)^+ + \sum_{s: a_s \neq 0} \sum_{t \in [l-1]} I_t (b_{ts})^+ + \sum_{s: u_s \neq 0} \sum_{t \in [l-1]} I_t (-v_{ts})^+ \right\},$$

where

$$\tilde{V}_l = \sum_{s:u_s \neq 0} v_{ls}, \qquad \tilde{B}_l = \sum_{s:a_s \neq 0} b_{ls}.$$

Then the maximum degree over all entries of M is bounded by $\max\{\mu_1, \mu_2\}$.

Proof. Let the first row of M correspond to the coefficient of $\mathbf{k}^{\mathbf{0}}$ in the common numerator of (6.7). We use $M_{\mathbf{p},j}$ to denote the polynomial in n multiplied by $a_j(n)\mathbf{k}^{\mathbf{p}}$ in the common numerator of (6.7). Then

$$\deg M_{\mathbf{p},j} \leq \deg M_{\mathbf{0},j} \quad \text{for all } j \in [J]_0 \text{ and } \mathbf{p}.$$

Let

$$\text{smb} = \sum_{s:a_s \neq 0} \max_{l \in [r]} b_{ls}^+, \qquad \text{and} \quad \text{smv} = \sum_{s:u_s \neq 0} \max_{l \in [r]} (-v_{ls})^+.$$

We know that

$$\begin{aligned}\deg M_{\mathbf{0},j}(n) = \deg_n P(n, \mathbf{k}) + \text{smb} + \text{smv} \\ + \sum_{s:a_s \neq 0} \Big(j(-a_s)^+ + (J-j)a_s^+ + \sum_{l \in [r]} I_l b_{ls}^+ \Big) \\ + \sum_{s:u_s \neq 0} \Big(j(u_s)^+ + (J-j)(-u_s)^+ + \sum_{l \in [r]} I_l (-v_{ls})^+ \Big).\end{aligned}$$

Therefore,

$$\max_{j \in [J]_0} \deg M_{\mathbf{0},j}(n) = \deg_n P(n, \mathbf{k}) + \mathbf{I} \cdot \tilde{\mathfrak{B}} + J(\tilde{A} + (\tilde{U} - \tilde{A})^+) + \text{smb} + \text{smv} = \mu_1,$$

where the variables $\tilde{\mathfrak{B}}$, $\tilde{\mathcal{A}}$, $\tilde{U}$ and $\tilde{A}$ are defined in the statement of Lemma 5.4.

For the remaining $M_{\mathbf{p},(i,\mathbf{e})}$, those multiplied by $c_i(\mathbf{e}, n)\mathbf{k}^{\mathbf{p}}$, we compute an upper bound for the maximum degree from the expressions of R_{i1} and R_{i2}, $(i \in [r])$. For all $\mathbf{e}$, $\mathbf{p}$, and

$i \in [r]$, we have

$$
\begin{aligned}
&\deg M_{\mathbf{p},(i,\mathbf{e})}(n)\\
&\leq \mathrm{smb} + \mathrm{smv}\\
&\quad + \max_l \Bigg\{ \sum_{s:a_s\neq 0}\Big(b_{ls}^+ + \sum_{t\in[l-1]} I_t b_{ts}^+\Big) + \sum_{s:u_s\neq 0}\Big((-v_{ls})^+ + \sum_{t\in[l-1]} I_t(-v_{ts})^+\Big)\quad ,\\
&\qquad\quad \sum_{s:a_s\neq 0}\Big((-b_{ls})^+ + \sum_{t\in[l-1]} I_t b_{ts}^+\Big) + \sum_{s:u_s\neq 0}\Big(v_{ls}^+ + \sum_{t\in[l-1]} I_t(-v_{ts})^+\Big) \Bigg\}\\
&= \mathrm{smb} + \mathrm{smv}\\
&\quad + \max_l \Big\{ \tilde{\mathcal{B}}_l + (\tilde{V}_l - \tilde{B}_l)^+ + \sum_{s:a_s\neq 0}\sum_{t\in[l-1]} I_t b_{ts}^+ + \sum_{s:u_s\neq 0}\sum_{t\in[l-1]} I_t(-v_{ts})^+ \Big\}\\
&= \mu_2.
\end{aligned}
$$

Thus the maximum degree over all entries of M is bounded by $\max\{\mu_1, \mu_2\}$. □

Step 2. An upper bound for the largest coefficient of the entries of M.

We estimate the largest coefficient of the entries of M by first finding an upper bound for $\max_{l,\mathbf{p},j\in[I]_0} |[n^l]M_{\mathbf{p},j}|$, then for $\max_{l,\mathbf{p},(i,\mathbf{e})} |[n^l]M_{\mathbf{p},(i,\mathbf{e})}|$, where $M_{\mathbf{p},j}$ is the polynomial in n multiplied by $a_j(n)\mathbf{k}^{\mathbf{p}}$, and respectively $M_{\mathbf{p},(i,\mathbf{e})}$ is the polynomial in n multiplied by $c_i(\mathbf{e}, n)\mathbf{k}^{\mathbf{p}}$ in the common numerator of (6.7).

Let

$$
\begin{aligned}
P(n,\mathbf{k}) &:= \sum_{\mathbf{l}=0}^{\mathbf{E}} \sum_{m=0}^{D} t_{\mathbf{l}m}\mathbf{k}^{\mathbf{l}} n^m,\\
\mu_3 &:= (1+J)^D \max\{|t_{\mathbf{l}m}| : \mathbf{l} \in [\mathbf{E}]_0 \text{ and } m \in [D]_0\},
\end{aligned}
$$

$$\mu_4 := \prod_{s:\mathbf{b}_s^+\neq 0} \prod_{i\in[\max_l (b_{ls})^+ -1]_0} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s - i|\}$$
$$\times \prod_{s:(-\mathbf{v}_s)^+\neq 0} \prod_{i\in[\max_l(-v_{ls})^+]} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + i|\}$$
$$\times \prod_{s:\mathbf{b}_s^+\neq 0} \prod_{i\in[\mathbf{I}\cdot(\mathbf{b}_s)^+ -1]_0} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s - J(a_s)^+ - i|\}$$
$$\times \prod_{s:(-\mathbf{v}_s)^+\neq 0} \prod_{i\in[\mathbf{I}\cdot(-\mathbf{v}_s)^+]} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + J(-u_s)^+ + i|\},$$
$$\mu_5 := \max_{j\in[J]_0}\Bigg(\prod_{s:a_s<0} \prod_{i\in[j(-a_s)^+]} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s + i|\}$$
$$\times \prod_{s:a_s>0} \prod_{i\in[(J-j)a_s^+]} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s - Ja_s^+ + i|\}$$
$$\times \prod_{s:u_s<0} \prod_{i\in[(J-j)(-u_s)^+]} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + J(-u_s)^+ + 1 - i|\}$$
$$\times \prod_{s:u_s>0} \prod_{i\in[j(u_s)^+]} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + 1 - i|\}\Bigg),$$

where $\prod_{i\in[x]_0} \max\{*\} = 1$ (resp. $\prod_{i\in[y]} \max\{*\} = 1$) in μ_4 and μ_5 if $x \notin \mathbb{N}_0$ (resp. $y \notin \mathbb{N}$), and

$$\pi_1 := J(\tilde{A} + (\tilde{U} - \tilde{A})^+) + \tilde{\mathbf{I}}\cdot\tilde{\mathfrak{B}} + \sum_{s\in[p]} \max_l (b_{ls})^+ + \sum_{s\in[q]} \max_l (-v_{ls})^+.$$

We know from Lemma 6.3 that

$$\frac{L_j}{a_j(n)} \preccurlyeq P(n-j,\mathbf{k})\mu_4\mu_5(n + \mathbf{k}\cdot\mathbf{1} + 1)^{\pi_1}.$$

Moreover, Lemma 6.2 states that

$$\max_{m,\mathbf{l}} \big|[n^m \mathbf{k}^{\mathbf{l}}]P(n-j,\mathbf{k})\big| \le \mu_3$$

for all $j \in [J]_0$. We therefore conclude that the largest coefficient of $M_{\mathbf{p},j}$ for all $\mathbf{p}$ and $j \in [J]_0$ is bounded above by $\mu_3\mu_4\mu_5(r+2)^{\pi_1}(D+1)\prod_{i\in[r]}(E_i+1)$.

We now compute an upper bound for the coefficients of $M_{\mathbf{p},(i,\mathbf{e})}$, for all exponents $\mathbf{p}$ and $\mathbf{e}$ of $\mathbf{k}$, and $i \in [r]$. Recall that $M_{\mathbf{p},(i,\mathbf{e})}$ is the polynomial in n multiplied by $c_i(\mathbf{e}, n)\mathbf{k}^{\mathbf{p}}$

in the common numerator of (6.7). After the expansion of R_{i1} and R_{i2}, we get for a fixed pair $(\mathbf{p}, i)$,

$$M_{\mathbf{p},(i,\mathbf{e})} = -\left([\mathbf{k}^{\mathbf{p}-\mathbf{e}}] \frac{R_{i1}}{\sum\limits_{\substack{0\leq \mathbf{e} \\ \mathbf{e}\cdot\mathbf{1}\leq \mathcal{N}_i}} c_i(\mathbf{e},n)\mathbf{k}^{\mathbf{e}}} \right) + \sum_{j\leq e_i} (-1)^j \binom{e_i}{j}$$
$$\times \left[\mathbf{k}^{\mathbf{p}-(e_1,\ldots,e_{i-1},e_i-j,e_{i+1},\ldots,e_r)}\right] \frac{R_{i2}}{\sum\limits_{\substack{0\leq \mathbf{e} \\ \mathbf{e}\cdot\mathbf{1}\leq \mathcal{N}_i}} \left(\prod\limits_{l:l\neq i} k_l^{e_l} \sum\limits_{j\leq e_i} (-1)^j c_i(\mathbf{e},n) \binom{e_i}{j} k_i^{e_i-j} \right)}.$$

Let

$$\mu_{i1} := \prod_{s:\mathbf{b}_s^+\neq 0} \left(\prod_{j\in[\max_{l\in[r]}(b_{ls})^+-1]_0} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s - j|\} \right.$$
$$\times \left. \prod_{j\in[(b_{is})^+ + \sum_{t\in[i-1]} I_t(b_{ts})^+]} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s - J(a_s)^+ - \mathbf{I}\cdot(\mathbf{b}_s^+) + j|\} \right)$$
$$\times \prod_{s:(-\mathbf{v}_s)^+\neq 0} \left(\prod_{j\in[\max_{l\in[r]}(-v_{ls})^+]} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + j|\} \right.$$
$$\times \left. \prod_{\substack{j\in[(-v_{is})^+ \\ + \sum_{t\in[i-1]} I_t(-v_{ts})^+ - 1]_0}} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + J(-u_s)^+ + \mathbf{I}\cdot(-\mathbf{v}_s)^+ - j|\} \right),$$

$$\begin{aligned}\mu_{i2} := &\prod_{s:(-b_{is})^+>0}\ \prod_{j\in[(-b_{is})^+]} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s + j|\}\\
&\times \prod_{s:(b_{is})^+>0}\ \prod_{j\in[(b_{is})^+-1]_0} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s - j|\}\\
&\times \prod_{s:(v_{is})^+>0}\ \prod_{j\in[(v_{is})^+-1]_0} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s - j|\}\\
&\times \prod_{s:(-v_{is})^+>0}\ \prod_{j\in[(-v_{is})^+]} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + j|\}\\
&\times \prod_{s:(\mathbf{b}_s)^+\neq 0} \Bigg(\prod_{\substack{j\in[-(b_{is})^+\\ +\max_{l\in[r]}(b_{ls})^+]}} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s - \max_{l\in[r]}(b_{ls})^+ + j|\}\\
&\times \prod_{j\in[\sum_{t\in[i-1]} I_t(b_{ts})^+-1]_0} \max\{|a_s|, |b_{ls}|_{l\in[r]}, |c_s - J(a_s)^+ - \mathbf{I}\cdot(\mathbf{b}_s)^+ - j|\}\Bigg)\\
&\times \prod_{s:(-\mathbf{v}_s)^+\neq 0} \Bigg(\prod_{\substack{j\in[-(-v_{is})^+\\ +\max_{l\in[r]}(-v_{ls})^+-1]_0}} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + \max_{l\in[r]}(-v_{ls})^+ - j|\}\\
&\times \prod_{j\in[\sum_{t\in[i-1]} I_t(-v_{ts})^+]} \max\{|u_s|, |v_{ls}|_{l\in[r]}, |w_s + J(-u_s)^+ + \mathbf{I}\cdot(-\mathbf{v}_s)^+ + j|\}\Bigg),\end{aligned}$$

where $\prod_{j\in[x]_0}\max\{*\} = 1$ (resp. $\prod_{j\in[y]}\max\{*\} = 1$) in μ_{i1} and μ_{i2} if $x \notin \mathrm{N}_0$ (resp. $y \notin \mathrm{N}$).

Furthermore, let

$$\begin{aligned}\pi_{i1} := &\Big(\sum_{s\in[p]} \max_l b_{ls}^+ + b_{is}^+ + \sum_{t\in[i-1]} I_t(b_{ts})^+\Big)\\
&+ \Big(\sum_{s\in[q]} \max_l(-v_{ls})^+ + (-v_{is})^+ + \sum_{t\in[i-1]} I_t(-v_{ts})^+\Big),\end{aligned}$$

and

$$\begin{aligned}\pi_{i2} := &\sum_{l,s}|b_{ls}| + \sum_{l,s}|v_{ls}|\\
&+ \Big(\sum_{s\in[p]} \max_l b_{ls}^+ - b_{is}^+ + \sum_{t\in[i-1]} I_t(b_{ts})^+\Big)\\
&+ \Big(\sum_{s\in[q]} \max_l(-v_{ls})^+ - (-v_{is})^+ + \sum_{t\in[i-1]} I_t(-v_{ts})^+\Big).\end{aligned}$$

By Lemma 6.3,

$$\max_{\mathbf{e},m}\left|[n^m\mathbf{k}^{\mathbf{e}}]\frac{R_{i1}}{\sum_{\substack{0\le\mathbf{e}\le(\mathcal{N}_i,\mathcal{N}_i,\dots,\mathcal{N}_i)\\ \mathbf{e}\cdot\mathbf{1}\le\mathcal{N}_i}} c_i(\mathbf{e},n)\mathbf{k}^{\mathbf{e}}}\right|$$

is bounded above by $(r+2)^{\pi_{i1}}\mu_{i1}$, and

$$\max_{\mathbf{e},m}\left|[n^m\mathbf{k}^{\mathbf{e}}]\frac{R_{i2}}{\sum_{\substack{0\le\mathbf{e}\le(\mathcal{N}_i,\mathcal{N}_i,\dots,\mathcal{N}_i)\\ \mathbf{e}\cdot\mathbf{1}\le\mathcal{N}_i}}\left(\prod_{l:l\ne i}k_l^{e_l}\sum_{j\le e_i}(-1)^j c_i(\mathbf{e},n)\binom{e_i}{j}k_i^{e_i-j}\right)}\right|$$

is bounded above by $(r+2)^{\pi_{i2}}\mu_{i2}$. Thus from the expressions of $M_{\mathbf{p},(i,\mathbf{e})}$, we conclude that for a fixed i, the largest coefficient of the entries of M multiplied by $c_i(\mathbf{e},n)$ is bounded above by

$$(r+2)^{\pi_{i1}}\mu_{i1}+2^{\mathcal{N}_i}(r+2)^{\pi_{i2}}\mu_{i2}.$$

Let

$$\mu_6:=\max\left\{\mu_3\mu_4\mu_5(r+2)^{\pi_1}(D+1)\prod_{i\in[r]}(E_i+1)\right\}\cup\left\{(r+2)^{\pi_{i1}}\mu_{i1}+2^{\mathcal{N}_i}(r+2)^{\pi_{i2}}\mu_{i2}\right\}_{i\in[r]}.$$

Then the largest coefficient of the entries of M is bounded above by μ_6. We formulate the result above in the following

Lemma 6.5. *The absolute value of the largest coefficient of the entries of M is bounded above by μ_6.*

Step 3. The size of M.

As in Chapter 5, we need the size of M to complete the estimate for the proof of Theorem 6.1.

Proposition 6.6. *Let*

$$A := \sum_{s:\mathbf{b}_s\neq \mathbf{0}} a_s, \qquad B_l := \sum_{s\in[p]} b_{ls}, \qquad U := \sum_{s:\mathbf{v}_s\neq\mathbf{0}} u_s, \qquad V_l := \sum_{s\in[q]} v_{ls},$$

$$\mathcal{A} := \sum_{s:\mathbf{b}_s\neq\mathbf{0}} a_s^+ + \sum_{s:\mathbf{v}_s\neq\mathbf{0}} (-u_s)^+, \qquad \mathcal{B}_l := \sum_{s\in[p]} b_{ls}^+ + \sum_{s\in[q]}(-v_{ls})^+,$$

$$\mathcal{N}_i := \deg_{\mathbf{k}} P(n,\mathbf{k}) + (I_i - 1)(\mathcal{B}_i + (V_i - B_i)^+)$$

$$+J(\mathcal{A} + (U-A)^+) + \sum_{i<t\leq r} I_t(\mathcal{B}_t + (V_t - B_t)^+).$$

Then the number, ν, of rows in M is at most

$$\mu_7 := 1+\deg_{\mathbf{k}} P(n,\mathbf{k})+\sum_{s\in[p]} \max_{l\in[r]} b_{ls}^+ + \sum_{s\in[q]} \max_{l\in[r]}(-v_{ls})^+ + J(\mathcal{A}+(U-A)^+)+\mathbf{I}\cdot(\mathbf{B}+(\mathbf{V}-\mathbf{B})^+),$$

and the number of columns in M is

$$\mu_8 := 1 + J + \sum_{i\in[r]} \sum_{l\in[\mathcal{N}_i]_0} \binom{l+r-1}{r-1}.$$

Proof. The number of rows in M is $1+$ the degree in $\mathbf{k}$ of the common numerator in (6.7). But the degree in $\mathbf{k}$ of the common numerator in (6.7) is less than or equal to

$$\max\left\{ \max_{j\in[J]_0} \deg_{\mathbf{k}} L_j, \quad \max_{i\in[r]} \deg_{\mathbf{k}} R_{i1}, \quad \max_{i\in[r]} \deg_{\mathbf{k}} R_{i2} \right\}.$$

From the expressions for L_j, R_{i1} and R_{i2}, we get that

$$\max_{j\in[J]_0} \deg_{\mathbf{k}} L_j = \max_{j\in[J]_0} \Bigg(\deg_{\mathbf{k}} P(n,\mathbf{k}) + \sum_{s\in[p]} \max_{l\in[r]} b_{ls}^+ + \sum_{s\in[q]} \max_{l\in[r]}(-v_{ls})^+$$

$$+ j\Big(\sum_{s\in[p]:\mathbf{b}_s\neq\mathbf{0}} (-a_s)^+ + \sum_{s\in[q]:\mathbf{v}_s\neq\mathbf{0}} u_s^+ \Big)$$

$$+ (J-j)\Big(\sum_{s\in[p]:\mathbf{b}_s\neq\mathbf{0}} a_s^+ + \sum_{s\in[q]:\mathbf{v}_s\neq\mathbf{0}} (-u_s)^+ \Big)$$

$$+ \sum_{s\in[p]} \mathbf{I}\cdot\mathbf{b}_s^+ + \sum_{s\in[q]} \mathbf{I}\cdot(-\mathbf{v}_s)^+ \Bigg)$$

$$= \deg_{\mathbf{k}} P(n,\mathbf{k}) + \sum_{s\in[p]} \max_{l\in[r]} b_{ls}^{+} + \sum_{s\in[q]} \max_{l\in[r]}(-v_{ls})^{+} + \mathbf{I}\cdot\mathcal{B} + J(\mathcal{A} + (U-A)^{+});$$

that

$$\begin{aligned}\max_{i\in[r]} \deg_{\mathbf{k}} R_{i1} \le \max_{i\in[r]}\Big(&\mathcal{N}_i + \sum_{s\in[p]} \max_{l\in[r]}(b_{ls})^{+} + \sum_{s\in[q]} \max_{l\in[r]}(-v_{ls})^{+} \\ &+ \sum_{s\in[p]}\Big(b_{is} + \sum_{t\in[i-1]} I_t(b_{st})^{+}\Big) + \sum_{s\in[q]}\Big((-v_{is})^{+} + \sum_{t\in[i-1]} I_t(-v_{ts})^{+}\Big)\Big) \\ = \deg_{\mathbf{k}} &P(n,\mathbf{k}) + J(\mathcal{A} + (U-A)^{+}) + \mathbf{I}\cdot\mathcal{B} \\ &+ \sum_{s\in[p]} \max_{l\in[r]}(b_{ls})^{+} + \sum_{s\in[q]} \max_{l\in[r]}(-v_{ls})^{+} \\ &+ \max_{i\in[r]}\Big(\sum_{i<t\le r} I_t(V_t - B_t)^{+} - (V_i - B_i)^{+}\Big),\end{aligned}$$

and that

$$\begin{aligned}\max_{i\in[r]} \deg_{\mathbf{k}} R_{i2} \le \max_{i\in[r]}\Big(&\mathcal{N}_i + \sum_{s\in[p]} |b_{is}| + \sum_{s\in[q]} |v_{is}| \\ &+ \sum_{s\in[p]}\Big(-b_{is}^{+} + \max_{l\in[r]} b_{ls}^{+}\Big) + \sum_{s\in[q]}\Big(-(-v_{is})^{+} + \max_{l\in[r]}(-v_{ls})^{+}\Big) \\ &+ \sum_{s\in[p]}\sum_{t\in[i-1]} I_t(b_{ts})^{+} + \sum_{s\in[q]}\sum_{t\in[i-1]} I_t(-v_{ts})^{+}\Big) \\ = \deg_{\mathbf{k}} &P(n,\mathbf{k}) + J(\mathcal{A} + (U-A)^{+}) + \mathbf{I}\cdot\mathcal{B} \\ &+ \sum_{s\in[p]} \max_{l\in[r]}(b_{ls})^{+} + \sum_{s\in[q]} \max_{l\in[r]}(-v_{ls})^{+} \\ &+ \max_{i\in[r]}\Big(\sum_{s\in[p]} |b_{is}| + \sum_{s\in[q]} |v_{is}| - 2\mathcal{B}_i - (V_i - B_i)^{+} + \sum_{i<t\le r} I_t(V_t - B_t)^{+}\Big).\end{aligned}$$

Thus

$$\begin{aligned}
\nu \le 1 + \deg_{\mathbf{k}} P(n,\mathbf{k}) &+ \sum_{s\in[p]} \max_{l\in[r]}(b_{ls})^+ + \sum_{s\in[q]} \max_{l\in[r]}(-v_{ls})^+ \\
&+ J(\mathcal{A} + (U - \mathcal{A})^+) + \mathbf{I}\cdot\mathcal{B} \\
&+ \max\left\{ \max_{i\in[r]}\Big(\sum_{i<t\le r} I_t(V_t - B_t)^+ - (V_i - B_i)^+\Big) \right., \\
&\qquad \left. \max_{i\in[r]}\Big(\sum_{i<t\le r} I_t(V_t - B_t)^+ - (B_i - V_i)^+\Big) \right\} \\
= 1 + \deg_{\mathbf{k}} P(n,\mathbf{k}) &+ \sum_{s\in[p]} \max_{l\in[r]}(b_{ls})^+ + \sum_{s\in[q]} \max_{l\in[r]}(-v_{ls})^+ \\
&+ J(\mathcal{A} + (U - \mathcal{A})^+) + \mathbf{I}\cdot(\mathcal{B} + (\mathbf{V} - \mathbf{B})^+) = \mu_7.
\end{aligned}$$

Next we show that M has $1 + J + \sum_{i\in[r]}\sum_{l=0}^{\mathcal{N}_i}\binom{l+r-1}{r-1}$ columns. We know that the number of columns in M is equal to the number of entries in the vector $\mathbf{x}$ which is

$$\big|\{a_j(n) : j \in [J]_0\} \cup \{c_i(\mathbf{e},n) : \mathbf{e} \ge \mathbf{0}, i \in [r]\}\big|.$$

There are $J+1$ $a_j(n)$'s. Our task is to find $|\{c_i(\mathbf{e},n)\}|$ for all $\mathbf{e} \ge \mathbf{0}$ such that $\mathbf{e}\cdot\mathbf{1} \le \mathcal{N}_i$ for all $i \in [r]$.

For a fixed $i \in [r]$, we note that $|\{c_i(\mathbf{e},n) : \mathbf{e} \ge \mathbf{0}, \mathbf{e}\cdot\mathbf{1} \le \mathcal{N}_i\}|$ is the number of ordered partitions of l ($l \in [\mathcal{N}_i]_0$) into r non-negative parts. In other words,

$$\big|\{c_i(\mathbf{e},n) : \mathbf{e} \ge \mathbf{0}, \mathbf{e}\cdot\mathbf{1} \le \mathcal{N}_i\}\big| = \sum_{l\in[\mathcal{N}_i]_0} [x^l]\frac{1}{(1-x)^r} = \sum_{l\in[\mathcal{N}_i]_0}\binom{l+r-1}{r-1}. \tag{6.8}$$

Summing (6.8) over all $i \in [r]$, we get

$$\big|\{c_i(\mathbf{e},n) : \mathbf{e} \ge \mathbf{0}, \mathbf{e}\cdot\mathbf{1} \le \mathcal{N}_i, i \in [r]\}\big| = \sum_{i\in[r]}\sum_{l\in[\mathcal{N}_i]_0}\binom{l+r-1}{r-1}.$$

Thus the number of columns in M is $\mu_8 = 1 + J + \sum_{i\in[r]}\sum_{l\in[\mathcal{N}_i]_0}\binom{l+r-1}{r-1}$.

Step 4. Upper bounds for $\deg\det M_1'$ *and* $\max_i |[n^i]\det M_1'|$.

Since the rank ρ of M is bounded by ν, the number of rows, $\deg\det M_i'$ ($i \in [\rho]$), or more specifically,

$$\deg\det M_1' \leq \nu \max\{\mu_1, \mu_2\} \leq \mu_7 \max\{\mu_1, \mu_2\}.$$

From the way we obtain M_1' from M (—see §5.3 for the steps), we have $\max_i |[n^i]M_1'|$ is less than the product of the number of columns of M and the largest coefficient of the entries of M. Thus

$$\max_i |[n^i]M_1'| < \mu_8\mu_6.$$

Using the definition of the determinant, we have

$$\max_i |[n^i]\det M_1'| \leq \mu_7!\,(\mu_8\mu_6(\max\{\mu_1,\mu_2\}+1))^{\mu_7}. \quad \square$$

In Stage 4, we apply Proposition 5.4 to get n_1, thus completing the proof. Recall from §5.4 that a polynomial solution for the leading coefficient, $a_0(n)$ is $\det M_1'$. From Step 4 of Stage 3, we have $\mu_7 \max\{\mu_1, \mu_2\}$ as an upper bound for the degree of $a_0(n)$, and $\mu_7!\,(\mu_8\mu_6(\max\{\mu_1,\mu_2\}+1))^{\mu_7}$ as an upper bound for the largest coefficient of $a_0(n)$. By Proposition 5.4, $a_0(n) \neq 0$ for all

$$n \geq \mu_7 \max\{\mu_1, \mu_2\}\mu_7!\,(\mu_8\mu_6(\max\{\mu_1,\mu_2\}+1))^{\mu_7} =: n_a.$$

Since the recurrence relation satisfied by $\sum_{\mathbf{k}} F(n, \mathbf{k})$ is of order at most J, we can take n_1 to be $n_a + J - 1$ when the given n_0 is small relative to n_a. On the other hand, if n_0 is large, it suffices to take n_1 to be

$$\max\{n_0 + \mu_7 \max\{\mu_1, \mu_2\}, n_a + J - 1\}. \quad \square$$

CHAPTER VII

WHEN IS $\sum_k F(n,k)$ HYPERGEOMETRIC?

Let $F(n,k)$ be a proper hypergeometric term. Suppose we want to know if $\sum_k F(n,k)$ is hypergeometric. Petkovšek's algorithm [P] tells us how to check if $\sum_k F(n,k)$ is hypergeometrically summable from the recurrence satisfied by $\sum_k F(n,k)$ provided $F(n,k)$ contains no parameters.

From Theorem 3.1 of [WZ3], we know the existence of polynomials, $\alpha_{i,j}(n)$, not all zero, and integers, I, J, such that

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}(n)F(n-j,k-i)=0. \tag{7.1}$$

Writing (7.1) in the following way

$$\sum_{i=0}^{I}\alpha_{i,0}(n)F(n,k-i)+\sum_{i=0}^{I}\alpha_{i,1}(n)F(n-1,k-i)+\cdots+\sum_{i=0}^{I}\alpha_{i,J}(n)F(n-J,k-i)=0, \tag{7.2}$$

and summing over k, we get for an admissible proper-hypergeometric term, $F(n,k)$,

$$\Big(\sum_{i=0}^{I}\alpha_{i,0}(n)\Big)f_n+\Big(\sum_{i=0}^{I}\alpha_{i,1}(n)\Big)f_{n-1}+\cdots+\Big(\sum_{i=0}^{I}\alpha_{i,J}(n)\Big)f_{n-J}=0, \tag{7.3}$$

where $f_n=\sum_k F(n,k)$. If there are rational functions, $\alpha_{i,j}(n)$, such that

$$\sum_{i=0}^{I}\alpha_{i,0}(n)=1 \qquad \text{and} \qquad \sum_{i=0}^{I}\alpha_{i,j}(n)=0 \quad (j\geq 2), \tag{7.4}$$

then f_n is certainly hypergeometric because

$$f_n=(\text{rational function in } n)f_{n-1}.$$

Remarks. We do not have to insist on (7.4). In fact, f_n is hypergeometric if there exist $\alpha_{i,j}(n)$ such that in (7.3) all but two consecutive coefficients of the f_n's are zero.

We present an algorithm that takes an admissible proper-hypergeometric term, $F(n,k)$, and checks if $\sum_k F(n,k)$ is hypergeometric by checking the sufficient condition.

ALGORITHM SUFF

Step 1. Let $\alpha_{i,j}$ be indeterminate polynomials in n. Form

$$\sum_{i,j}^{I,J} \frac{\alpha_{i,j}(n)F(n-j,k-i)}{F(n,k)} = 0. \tag{7.5}$$

Step 2. Find a common denominator of (7.5), and put everything over the common denominator. From the degree in k of the common denominator, find I, J such that

$$(I+1)(J+1) \geq 2 + J + \deg_k \text{Numerator}.$$

Step 3. Solve for $\alpha_{i,j}(n)$ in the system of homogeneous linear equations obtained from the numerator by setting the coefficient of each power of k to zero. Let M be a matrix over $\mathbb{Z}[n]$ such that $M\boldsymbol{\alpha} = \mathbf{0}$, where

$$\boldsymbol{\alpha} = (\alpha_{0,0}, \alpha_{1,0}, \ldots, \alpha_{I,0}, \alpha_{1,0}, \ldots, \alpha_{I,J})^{\mathrm{t}}$$

and the ith row of M corresponds to the coefficients of k^{i-1}.

Step 4. To incorporate (7.4), we augment M by adjoining C to the bottom of M, where C is the matrix corresponding to (7.4) with the condition for $\sum_{i=0}^{I} \alpha_{i,0} = 1$ in the last row of C. Note that C is J by $(I+1)(J+1)$. Let the augmented matrix be $A := \begin{bmatrix} M \\ C \end{bmatrix}$. We want to solve for $\boldsymbol{\alpha}$ such that $A\boldsymbol{\alpha} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Step 5. To see if such an α exists, we devote the rest of the algorithm to checking whether

$$\operatorname{rank} A = \operatorname{rank} \begin{bmatrix} A & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \end{bmatrix},$$

since α exists iff $\operatorname{rank} A = \operatorname{rank} \begin{bmatrix} A & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \end{bmatrix}$.

Step 6. Fix the last row of A and row reduce the rest of A. Attach the last vector of the elementary basis to the resulting matrix.

Step 7. To see if

$$\operatorname{rank} A = \operatorname{rank} \begin{bmatrix} A & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \end{bmatrix},$$

we perform Gaussian elimination to the last row of the matrix from Step 6. If at any time, we get a row whose first (not also the last) non-zero entry cannot be eliminated, then we are done because

$$\operatorname{rank} A = \operatorname{rank} \begin{bmatrix} A & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \end{bmatrix}$$

and such an α exists. Otherwise all but the last entry in the last row survives the process. In this case,

$$\operatorname{rank} A \neq \operatorname{rank} \begin{bmatrix} & 0 \\ A & \vdots \\ & 0 \\ & 1 \end{bmatrix},$$

and no α exists.

The algorithm above checks if a given admissible proper-hypergeometric term satisfies the sufficient condition (7.4). If it does, then the sum is hypergeometric. Petkovšek [P] gives necessary conditions for $\sum_k F(n,k)$ to be hypergeometric by solving the following decision problem.

Given a linear recurrence relation of order h with polynomial coefficients, decide whether the recurrence has a solution that satisfies another recurrence of order 1*; and if so, find that recurrence of order* 1.

His algorithm works if the polynomial coefficients do not contain any parameters. We still do not know any necessary conditions on an admissible proper-hypergeometric term, $F(n,k)$, for the sum $\sum_k F(n,k)$ to be hypergeometric.

BIBLIOGRAPHY

[An1] G. E. Andrews, *Applications of basic hypergeometric functions*, SIAM Review **16** (1974), 441–484.

[An2] ———, *Problems and properties for basic hypergeometric functions*, Theory and Applications of Special Functions (R. Askey, ed.), Academic Press, New York, 1975, pp. 191–224.

[An3] ———, *On q-analogues of the Watson and Whipple summations*, SIAM J. Math. Anal. **7** (1976), 332–336.

[An4] ———, *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications, vol. 2 (G. C. Rota, ed.), Addison Wesley, Reading, 1976.

[An5] ———, *q-Series: Their Development and Applications in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS series, vol. 66, Amer. Math. Soc., Providence, 1986.

[An6] ———, *Notes on the Dyson conjecture.*

[Ap] R. Apéry, *Irrationalité de* $\zeta(3)$, Asterisque **61** (1979), 11–13.

[As1] R. A. Askey, *Orthogonal Polynomials and Special Functions*, Regional Conference Series in Applied Mathematics, vol. 21, SIAM, 1975.

[As2] ———, *Some basic hypergeometric extensions of integrals of Selberg and Andrews*, SIAM J. Math. Anal. **11** (1980), no. 6, 938–951.

[As3] ———, *Special functions: Group theoretical aspects and applications (preface)* (1984), D. Reidel Publ. Co., Dordrecht.

[B] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Math. Tract No. 32, Cambridge University Press, London and New York, 1935; reprinted, Hafner, New York, 1964.

[deB] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.

[E] L. Euler, *Introductio in Analysis Infinitorum*, M-M Bousquet, Lausanne.

[F1] M. C. Fasenmyer, Ph. D. thesis, University of Michigan, 1945.

[F2] ———, *Some generalized hypergeometric polynomials*, Bull. Amer. Math. Soc. **53** (1947), 806–812.

[F3] ———, *A note on pure recurrence relations*, Amer. Math. Monthly **56** (1949), 14–17.

[Fo1] D. Foata, *A combinatorial proof of the Mehler formula*, J. Comb. Theory, Ser. A **24** (1978), 250–259.

[Fo2] ———, *Combinatoire des identités sur les polynomes orthogonaux*, Proc. ICM 83, Varsovie (1983), 1541–1553.

[Gau] C. F. Gauß, *Disquisitiones generales circa seriem infinitam*

$$1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+\cdots$$

Pars prior, Thesis delivered to the Royal Society in Göttingen, 20 January 1812, Commentationes societaris regiæ scientiarum Gottingenis recentiores **2** (1813), Reprinted in his *Werke*, volume 3, 123–163, together with an unpublished sequel on pages 207–229.

[GR] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and Its Applications, Cambridge Univ. Press, Cambridge, 1990.

[GS] L. Gessel and D. Stanton, *Short proofs of Saalschütz' and Dixon's theorems*, J. Comb. Theory, Ser. A **38** (1985), 87–90.

[Go] I. J. Good, *Short proof of a conjecture by Dyson*, J. Math. Phys. **11** (1970), 1884.

[G] R. W. Gosper, *Decision procedure for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA **75** (1978), 40–42.

[GKP] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison–Wesley, Reading, Massachusetts, 1989.

[H] T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics, 73, Springer, New York, 1989.

[Ma] I. G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal. **13** (1982), 988–1007.

[Mi] W. Miller, *Symmetry and separation of variables*, Encyclopedia Math. Appl., vol. 4, Addison–Wesley, London, 1977.

[M] S. C. Milne, *A q-analog of the Gauss summation theorem for hypergeometric series in $U(n)$*, Adv. Math. **72** (1988), 59–131.

[MP] C. B. Morrey and M. H. Protter, *A First Course in Real Analysis*, Undergraduate Texts in Mathematics, Springer, New York, 1977.

[Pf] J. F. Pfaff, *Observationes analyticæ ad L. Euleri institutiones calculi integralis, Vol. IV, Supplem. II & IV*, Nova acta academiæ scientiarum Petropolitanæ **11**, Histore section, 37–57, This volume, printed in 1798, contains mostly proceedings from 1793, although Pfaff's memoir was actually received in 1797.

[P] M. Petkovšek, *Finding closed-form solutions of difference equations by symbolic methods*, Ph. D. thesis, Carnegie Mellon University (1991).

[PS] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 216, vol. 2, Springer, New York, 76.

[PBM] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and series*, translated from Russian by G. G. Gould, vol. 3, Gordon and Breach, New York, 1990.

[R] E. D. Rainville, *Special functions*, Macmillan, New York, 1960; reprinted, Chelsea, Bronx, NY, 1971.

[S] W. W. Sawyer, *Prelude to Mathematics*, Penguin, Baltimore, 1955.

[Sz] G. Szegö, *Orthogonal Polynomials*, AMS Colloquium Publications, vol. 23, Amer. Math. Soc., New York, 1959.

[V] P. Verbaeten, *The automatic construction of pure recurrence relations*, Proc. EUROSAM '74, ACM-SIGSAM Bulletin **8** (1974), 96–98.

[W1] H. S. Wilf, *Perron–Frobenius theory and the zeros of polynomials*, Proc. Amer. Math. Soc. **12** (1961), no. 2, 247–250.

[W2] ———, *Sums of closed form functions satisfy recurrence relations*, Private notes (1991), 1–4.

[WZ1] H. S. Wilf and D. Zeilberger, *Towards computerized proofs of identities*, Bull. Amer. Math. Soc. **23** (1990), 77–83.

[WZ2] ———, *Rational functions certify combinatorial identities*, J. Amer. Math. Soc. **3** (1990), 147–158.

[WZ3] ———, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. **108** (1992), 575–633.

[WZ4] ———, *Rational function certification of multisum/integral/"q" identities*, Bull. Amer. Math. Soc. **27** (1992), 148–153.

[Wi] K. Wilson, *Proof of a conjecture by Dyson*, J. Math. Phys. **3** (1962), 1040–1043.

[Z1] D. Zeilberger, *All binomial identities are verifiable*, Proc. Natl. Acad. Sci. USA **78** (1981), no. 7, 4000.

[Z2] ———, *Sister Celine's Technique and Its Generalizations*, J. of Mathematical analysis and applications **85** (1982), no. 1, 114–145.

[Z3] ———, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. **80** (1990), 207–211.

[Z4] ———, *A holonomic systems approach to special functions identities*, J. of Computational and Applied Mathematics **32** (1990), 321–368.

[Z5] ———, *The method of creative telescoping*, J. Symbolic Computation **11** (1991), 195–204.

INDICES

Symbols

TERMINOLOGY