PROBLEMS IN RESTRICTED PARTITIONS

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ABSTRACT. This note contains discussion on problems concerning partitions whose parts are powers of a given base b.

1. NOTATION

Let $p_b(n)$ be the number of b-ary partitions of n. Let $B_b(q)$ be the generating function of $p_b(n)$, that is,

$$B_b(q) = \sum_{n=0}^{\infty} p_b(n)q^n = \prod_{i=0}^{\infty} \frac{1}{1-q^{b^i}}.$$

Abstractify this slightly:

$$B_b(q,m) = \sum_{n \geqslant 0} p_b(b^m n)q^n.$$

Let Δ denote the finite difference operator: $\Delta_u = f(u+1) - f(u)$.

2. PREVIOUS WORK

Theorem 2.1. *The b-ary parition function satisfies the following recurrence relation:*

$$p_b(n) = 0 \textit{ for } n < 0$$

$$p_b(n) = 1 \textit{ for } n \in b$$

(**RI**) $$p_b(bn+i) = p_b(bn) \textit{ for } i \in b$$

(**RII**) $$p_b(bn) = p_b(bn-1) + p_b(n)$$

Combining **RI** and **RII** reveals

(**RII***) $$p_b(bn) = p_b(bn-b) + p_b(n).$$

Theorem 2.2 (Rødseth, Sellers)**.** *Let $r \geqslant 1$ and suppose that σ_r can be expressed as*

$$\sigma_r = \sum_{i=2}^{r} \varepsilon_i b^i$$

where $\varepsilon_i \in 2$ for each i. Finally, let $c_r = 2^{[\![b \text{ odd}]\!](r-1)}$. Then for all $n \geqslant 1$,

$$p_b(b^{r+1}n - \sigma_r - b) \in \frac{b^r}{c_r}\mathbb{Z}.$$

Corollary 2.3. *Assume a base b, a length r and an integer $m \in 2^r$ are given. Define $c(r,b) = 2^{[\![b \text{ odd}]\!](r-1)}$, and let $s = (\lfloor \frac{m}{2^i} \rfloor \,\%\, b)_{i<\omega}$. For all n*

$$p_b(b^{r+1}n - \nu_b(s) - b) \in \frac{b^r}{c(r,b)}\mathbb{Z}.$$

3. Facts

$$B_b(q) = B_b(q^{b^m}) \prod_{i \in m} \frac{1}{1 - q^{b^i}}$$
$$B_b(q^b) = (1-q)B_b(q)$$

Let ζ_m be a primitive mth root of unity.

$$B_b(m,q) = \sum_{n=0}^{\infty} p_b(b^m n) q^n$$

$$B_b(0,q^b) = (1-q)B_b(0,q) = \sum_{n \in \mathbb{Z}} p_b(n) q^{nb}$$
$$\mathrm{Re}((\zeta_{b^m} q^{b^{-m}})^{-c} B_b(0, q^{b^{-m}} \zeta_{b^m})) = \sum_{n \in \mathbb{Z}} p_b(b^m n + c) q^n$$

Lemma 3.1. *For all $n, m \geqslant 1$ and $1 \leqslant k < b^m$ there exist polynomials $g_j(x)$ of degree j for $j \in m$ such that*

$$p_b(b^m n + k) = p_b(b^m n) + \sum_{j=1}^{m-1} g_j(b) p_b(b^j n). \qquad (\mathrm{IH}(m))$$

Proof. The proof proceeds by induction on m. For $m = 1$ the induction hypothesis says $p_b(n+k) = p_b(n)$ for $1 \leqslant k < b$ which is true by **RI**. Assume that $\mathrm{IH}(m')$ is true for all $m' < m$. Let $k = ub + v$. It may be assumed that $v = 0$ because if $v > 0$ then by **RI**

$$p_b(b^m n + k) = p_b(b^m n + ub + v) = p_b(b^m n + ub).$$

Therefore, applying **RII*** a total of u times:

$$\begin{aligned}
p_b(b^m n + ub) &= p_b(b^m n + (u-1)b) + p_b(b^{m-1}n + u) \text{ (\textbf{RII*} applied once)} \\
&= p_b(b^m n + (u-2)b) + p_b(b^{m-1}n + u - 1) + p_b(b^{m-1}n + u) \text{ (twice)} \\
&= p_b(b^m n + (u-2)b) + \sum_{j=0}^{1} p_b(b^{m-1}n + u - j) \text{ then after } u \text{ times} \\
&= p_b(b^m n) + \sum_{j=0}^{u-1} p_b(b^{m-1}n + u - j) \text{ let } k = u - j \\
&= p_b(b^m n) + \sum_{k=1}^{u} p_b(b^{m-1}n + k) \text{ then by IH}(m-1) \\
&= p_b(b^m n) + \sum_{k=1}^{u} \left(p_b(b^{m-1}n) + \sum_{l=1}^{m-2} g_{k,l}(b) p_b(b^l n) \right) \\
&= p_b(b^m n) + u p_b(b^{m-1}n) + \sum_{k=1}^{u} \sum_{l=1}^{m-2} g_{k,l}(b) p_b(b^l n) \\
&= p_b(b^m n) + u p_b(b^{m-1}n) + \sum_{l=1}^{m-2} \sum_{k=1}^{u} g_{k,l}(b) p_b(b^l n) \\
&= p_b(b^m n) + u p_b(b^{m-1}n) + \sum_{l=1}^{m-2} \left(\sum_{k=1}^{u} g_{k,l}(b) \right) p_b(b^l n)
\end{aligned}$$

Let $w = b^{m-1} - u$, $g_{m-1}(x) = x^{m-1} - w$ and $g_j(x) = \left(\sum_{k=0}^{u-1} g_{k,j}(x)\right)$ for $1 \leqslant j \leqslant m-2$. Then $u = b^{m-1} - w$ and $g_{m-1}(b) = b^{m-1} - w = u$, therefore

$$p_b(b^m n + ub + v) = p_b(b^m n) + \sum_{j=1}^{m-1} g_j(b) p_b(b^j n)$$

as stated by the theorem. ☐

Theorem 3.2. *For all m there exists a polynomial $f_m(x, q)$ of degree $m-1$ in q and degree $\binom{m}{2}$ in x such that*

$$f_m(b, q) B_b(0, q) = (1-q)^m B_b(m, q).$$

Proof. The proof proceeds by induction on m. The base case $m = 0$ is trivial, $f_0(x, q) = 1$. Assume that the theorem holds for all $m' < m$. Applying **RII*** to

$p_b(b^m n)$ yields the following:

$$\begin{aligned} p_b(b^m n) &= p_b(b^m n - b) + p_b(b^{m-1}n) \\ &= p_b(b^m(n-1) + b^m - b) + p_b(b^{m-1}n) \text{ and by Lemma 3.1} \\ &= p_b(b^m(n-1)) + \left(\sum_{j=1}^{m-1} g_j(b) p_b(b^j(n-1))\right) + p_b(b^{m-1}n) \end{aligned}$$

Then multiplying by q^n on both sides and summing:

$$\begin{aligned} B_b(m,q) &= \sum_{n\in\mathbb{Z}} p_b(b^m n) q^n \\ &= \sum_{n\in\mathbb{Z}} p_b(b^m(n-1))q^n + \sum_{n\in\mathbb{Z}} p_b(b^{m-1}n)q^n + \sum_{n\in\mathbb{Z}} \sum_{j=1}^{m-1} g_j(b) p_b(b^j(n-1))q^n \\ &= q\sum_{n\in\mathbb{Z}} p_b(b^m n)q^n + \sum_{n\in\mathbb{Z}} p_b(b^{m-1}n)q^n + q\sum_{n\in\mathbb{Z}} \sum_{j=1}^{m-1} g_j(b) p_b(b^j n)q^n \\ &= qB_b(m,q) + (1 + g_{m-1}(b)q) \sum_{n\in\mathbb{Z}} p_b(b^{m-1}n)q^n + q\sum_{j=1}^{m-2} g_j(b) \sum_{n\in\mathbb{Z}} p_b(b^j n)q^n \\ &= qB_b(m,q) + (1 + g_{m-1}(b)q) B_b(m-1,q) + q\sum_{j=1}^{m-2} g_j(b) B_b(j,q) \end{aligned}$$

By the induction hypothesis $(1-q)^{m-1}B_b(j,q) = B_b(0,q)(1-q)^{m-j-1}f_j(q)$. Then

$$\begin{aligned} (1-q)^m B_b(m,q) &= (1 + g_{m-1}(b)q) f_{m-1}(q) B_b(0,q) + q\sum_{j=1}^{m-2} g_j(b)(1-q)^{m-j-1} f_j(q) B_b(0,q) \\ &= \left((1 + g_{m-1}(b)q) f_{m-1}(q) + q\sum_{j=1}^{m-2} g_j(b)(1-q)^{m-j-1} f_j(q)\right) B_b(0,q) \end{aligned}$$

Consequently

$$f_m(x,q) = \left((1 + g_{m-1}(x)q) f_{m-1}(x,q) + q\sum_{j=1}^{m-2} g_j(x)(1-q)^{m-j-1} f_j(x,q)\right)$$

which is a polynomial of degree $m-1$ in q and degree $\binom{m}{2}$ in x, therefore

$$f_m(b,q)B_b(0,q) = (1-q)^m B_b(m,q)$$

which proves the theorem. □

Here is a table corresponding to identities of the form:

$$f_m(q)B_b(0,q) = (1-q)^m \sum p_b(b^m n)q^n$$

b	$f_3(b,q)$	$f_4(b,q)$
2	$1+6q+q^2$	$1+31q+31q^2+q^3$
3	$1+19q+7q^2$	$1+234q+447q^2+47q^3$
4	$1+42q+21q^2$	$1+1081q+2635q^2+379q^3$
5	$1+78q+46q^2$	$1+3072q+10218q^2+1704q^3$
6	$1+130q+85q^2$	$1+10335q+30735q^2+5585q^3$
7	$1+201q+141q^2$	$1+24896q+77801q^2+14951q^3$
8	$1+294q+217q^2$	$1+53669q+173747q^2+34727q^3$

b	$f_5(b,q)$
2	$1+196q+630q^2+196q^2+q^4$
3	$1+5822q+33504q^2+19040q^3+682q^4$
4	$1+79320q+561714q^2+387600q^3+19941q^4$
5	$1+642451q+5055891q^2+3835861q^3+231421q^4$
6	$1+3649340q+30621390q^2+24573740q^3+1621705q^4$
7	$1+16077981q+140871555q^2+117324441q^3+8201271q^4$
8	$1+58573732q+529473294q^2+452753140q^3+32941657q^4$
9	$1+184174970q+1704597594q^2+1486613030q^3+111398806q^4$
10	$1+515009556q+4855552326q^2+4299866676q^3+329571441q^4$
11	$1+1308822280q+12524820930q^2+11227696630q^3+876084760q^4$
12	$1+3072329216q+29763241530q^2+26948358536q^3+2133434941q^4$

and in general, these identities are proven below:

m	$f_m(b,q)$
1	1
2	$1+(b-1)q$
3	$1+\frac{1}{2}(b-1)\left((b^2+2b+4)q+(b^2-2)q^2\right)$
4	$1+\frac{1}{12}(b-1)\big((36+24b+18b^2+9b^3+5b^4+2b^5)q$ $+(-36-12b+6b^3+8b^4+8b^5)q^2$ $+(12-6b^2-3b^3-b^4+2b^5)q^3\big)$

4. Theorems

Theorem 4.1.

$$B_b(0,q)=(1-q)B_b(1,q)$$

Proof. By **RII***

$$p_b(bn)=p_b(b(n-1))+p_b(n)$$

so

$$p_b(n)=p_b(bn)-p_b(b(n-1))$$

so multiplying by q^n on both sides and summing over all integers n

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} p_b(n)q^n &= \sum_{n\in\mathbb{Z}} p_b(bn)q^n - \sum_{n\in\mathbb{Z}} p_b(b(n-1))q^n \\
&= \sum_{n\in\mathbb{Z}} p_b(bn)q^n - q\sum_{n\in\mathbb{Z}} p_b(bn)q^n \\
&= (1-q)\sum_{n\in\mathbb{Z}} p_b(bn)q^n \\
B_b(0,q) &= (1-q)B_b(1,q)
\end{aligned}$$

which is the statement of the theorem. □

Theorem 4.2.

$$(1+(b-1)q)B_b(0,q) = (1-q)^2 B_b(2,q)$$

Proof. Again appropriate iteration of the recursion rule **RII*** yields the result.

$$\begin{aligned}
p_b(b^2n) &= p_b(bn) + p_b(b(bn-1)) \text{ by } \mathbf{RII^*} \\
&= p_b(bn) + p_b(b(n-1)) + p_b(b(bn-2)) \text{ by } \mathbf{RII^*} \text{ applied twice,} \\
&\text{and after applying } \mathbf{RII^*} \text{ a total of } b-1 \text{ times} \\
&= p_b(bn) + (b-1)p_b(b(n-1)) + p_b(b(bn-b)) \\
&= p_b(bn) + (b-1)p_b(b(n-1)) + p_b(b^2(n-1)).
\end{aligned}$$

By passing to generating functions, the result is achieved.

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} p_b(b^2n)q^n &= \sum_{n\in\mathbb{Z}} p_b(bn)q^n + \sum_{n\in\mathbb{Z}}(b-1)p_b(b(n-1))q^n + \sum_{n\in\mathbb{Z}} p_b(b^2(n-1))q^n \\
B_b(2,q) &= B_b(1,q) + (b-1)qB_b(1,q) + qB_b(2,q) \\
(1-q)B_b(2,q) &= (1+(b-1)q)B_b(1,q) \\
(1-q)B_b(2,q) &= (1+(b-1)q)(1-q)^{-1}B_b(q).
\end{aligned}$$

Therefore

$$(1+(b-1)q)B_b(0,q) = (1-q)^2 B_b(2,q)$$

as stated by the theorem. □

$$
\begin{aligned}
p_b(b^3n) &= p_b(b^3n-b)+p_b(b^2n)\\
&= p_b(b^3n-2b)+p_b(b^2n-1)+p_b(b^2n)\\
&= p_b(b^3n-2b)+\sum_{j=0}^{1}p_b(b^2n-j)\\
&= \cdots\\
&= p_b(b^3(n-1))+\sum_{j=0}^{b^2-1}p_b(b^2n-j) \text{ let } k=b^2-j\\
&= p_b(b^3(n-1))+\sum_{k=1}^{b^2}p_b(b^2(n-1)+k)\\
&= p_b(b^3(n-1))+p_b(b^2n)+\sum_{k=1}^{b^2-1}p_b(b^2(n-1)+k) \text{ let } k=ub+v\\
&= p_b(b^3(n-1))+p_b(b^2n)+\sum_{v=1}^{b-1}p_b(b^2(n-1)+v)+\sum_{u=1}^{b-1}\sum_{v=0}^{b-1}p_b(b^2(n-1)+ub+v) \text{ by } \mathbf{RI}\\
&= p_b(b^3(n-1))+p_b(b^2n)+(b-1)p_b(b^2(n-1)+\sum_{u=1}^{b-1}\sum_{v=0}^{b-1}p_b(b^2(n-1)+ub)\\
&= p_b(b^3(n-1))+p_b(b^2n)+(b-1)p_b(b^2(n-1)+b\sum_{u=1}^{b-1}p_b(b^2(n-1)+ub)
\end{aligned}
$$

$$\begin{aligned}
\sum_{u=1}^{b-1} p_b(b^2(n-1)+ub) &= \sum_{u=1}^{b-1} p_b(b^2(n-1)+(u-1)b)+p_b(b(n-1)+u) \\
&= \sum_{u=1}^{b-1} p_b(b^2(n-1)+(u-2)b)+p_b(b(n-1)+(u-1))+p_b(b(n-1)+u) \\
&= \sum_{u=1}^{b-1} p_b(b^2(n-1)+(u-2)b)+\sum_{j=0}^{1} p_b(b(n-1)+u-j) \\
&= \cdots \\
&= \sum_{u=1}^{b-1} p_b(b^2(n-1))+\sum_{j=0}^{u-1} p_b(b(n-1)+u-j) \\
&= (b-1)p_b(b^2(n-1))+\sum_{u=1}^{b-1}\sum_{j=0}^{u-1} p_b(b(n-1)+u-j) \text{ let } k=u-j \\
&= (b-1)p_b(b^2(n-1))+\sum_{u=1}^{b-1}\sum_{k=1}^{u} p_b(b(n-1)+k) \text{ by } \mathbf{RI} \\
&= (b-1)p_b(b^2(n-1))+\sum_{u=1}^{b-1}\sum_{k=1}^{u} p_b(b(n-1)) \\
&= (b-1)p_b(b^2(n-1))+p_b(b(n-1))\sum_{u=1}^{b-1}\sum_{k=1}^{u} 1 \\
&= (b-1)p_b(b^2(n-1))+p_b(b(n-1))\sum_{u=1}^{b-1} u \\
&= (b-1)p_b(b^2(n-1))+\binom{b}{2}p_b(b(n-1))
\end{aligned}$$

Lemma 4.3.

$$\sum_{k=1}^{b-1} p_b(b^2n-kb) = (b-1)p_b(b^2(n-1))+\binom{b}{2}p_b(b(n-1))$$

Proof. Note that when $1 < k < b$

$$\begin{aligned}
p_b(b^2n-kb) &= p_b(b^2n-(k+1)b)+p_b(bn-k) \\
&= p_b(b^2n-(k+1)b)+p_b(bn-\left\lfloor \frac{k}{b} \right\rfloor b) \\
&= p_b(b^2n-(k+1)b)+p_b(bn-b) \\
&= p_b(b^2n-(k+1)b)+p_b(bn-bb^0) \\
&= p_b(b^2n-(k+1)b)+p_b(b(n-1))
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{k=1}^{b-1} p_b(b^2n - kb) &= \sum_{k=1}^{b-1}(p_b(b^2n-(k+1)b) + p_b(b(n-1))) \text{ (descend)} \\
&= (b-1)p_b(b(n-1)) + \sum_{k=1}^{b-1} p_b(b^2n-(k+1)b) \text{ (pop)} \\
&= (b-1)p_b(b(n-1)) + p_b(b^2n-b^2) + \sum_{k=1}^{b-2} p_b(b^2n-(k+1)b) \\
&= (b-1)p_b(b(n-1)) + p_b(b^2(n-1)) + \sum_{k=1}^{b-2} p_b(b^2n-(k+1)b) \text{ (reindex)} \\
&= (b-1)p_b(b(n-1)) + p_b(b^2(n-1)) + \sum_{k=2}^{b-1} p_b(b^2n-kb) \text{ (descend, pop, reindex)} \\
&= ((b-1)+(b-2))p_b(b(n-1)) + 2p_b(b^2(n-1)) + \sum_{k=3}^{b-1} p_b(b^2n-kb) \\
&= \cdots \text{descend, pop, reindex a total of } b-1 \text{ times} \\
&= \binom{b}{2} p_b(b(n-1)) + (b-1)p_b(b^2(n-1)) \\
&= (b-1)p_b(b^2(n-1)) + \binom{b}{2} p_b(b(n-1))
\end{aligned}$$

□

Definition 4.4 (The recurrence applied l times)**.**

$$F^l(p_b(b^m n)) = p_b(b^m n - bl) + \sum_{k=0}^{l-1} p_b(b^{m-1}n - k)$$

Theorem 4.5.

$$f_3(b,q)B_b(q) = (1-q)^3 B_b(q, b^3)$$

where

$$f_3(b,q) = 1 + \tfrac{1}{2}(b-1)\left((b^2+2b+4)q + (b^2-2)q^2\right)$$

Proof.

$$
\begin{aligned}
p_b(b^3n) &= p_b(b^3n-b) + p_b(b^2n) = F(p_b(b^3n)) \\
&= p_b(b^3n-b^3) + \sum_{k=0}^{b^2-1} p_b(b^2n-k) = F^{b^2}(p_b(b^3n)) \\
&= p_b(b^3(n-1)) + \sum_{k=0}^{b^2-1} p_b(b^2n-k) \\
&= p_b(b^3(n-1)) + \sum_{k=0}^{b^2-1} p_b(b^2n - \left\lceil \frac{k}{b} \right\rceil b) \\
&= p_b(b^3(n-1)) + p_b(b^2n) + \sum_{k=1}^{b^2-1} p_b(b^2n - \left\lceil \frac{k}{b} \right\rceil b) \\
&= p_b(b^3(n-1)) + p_b(b^2n) + \sum_{k=1}^{(b-1)(b+1)} p_b(b^2n - \left\lceil \frac{k}{b} \right\rceil b) \\
&= p_b(b^3(n-1)) + p_b(b^2n) + (b-1)p_b(b^2n-b^2) + \sum_{k=1}^{(b-1)(b)} p_b(b^2n - \left\lceil \frac{k}{b} \right\rceil b) \\
&= p_b(b^3(n-1)) + p_b(b^2n) + (b-1)p_b(b^2(n-1)) + \sum_{k=1}^{(b-1)(b)} p_b(b^2n - \left\lceil \frac{k}{b} \right\rceil b) \\
&= p_b(b^3(n-1)) + p_b(b^2n) + (b-1)p_b(b^2(n-1)) + b\sum_{k=1}^{b-1} p_b(b^2n-kb) \\
&\qquad \text{(so by Lemma 4.3)} \\
&= p_b(b^3(n-1)) + p_b(b^2n) + (b-1)p_b(b^2(n-1)) + b\left((b-1)p_b(b^2(n-1)) + \binom{b}{2}p_b(b(n-1))\right) \\
p_b(b^3n) &= p_b(b^3(n-1)) + p_b(b^2n) + (b+1)(b-1)p_b(b^2(n-1)) + b\binom{b}{2}p_b(b(n-1))
\end{aligned}
$$

Finally, upon passing to generating functions:

$$
\begin{aligned}
B_b(3,q) &= qB_b(3,q) + B_b(2,q) + (b+1)(b-1)qB_b(2,q) + b\binom{b}{2}qB_b(1,q) \\
(1-q)B_b(3,q) &= (1+(b+1)(b-1)q)B_b(2,q) + b\binom{b}{2}qB_b(1,q) \\
(1-q)B_b(3,q) &= (1+(b+1)(b-1)q)\frac{1+(b-1)q}{(1-q)^2}B_b(q) + b\binom{b}{2}q(1-q)^{-1}B_b(q) \\
(1-q)^3B_b(3,q) &= (1+(b+1)(b-1)q)(1+(b-1))qB_b(q) + b\binom{b}{2}q(1-q)B_b(q) \\
(1-q)^3B_b(3,q) &= \left(1+\left((b^2-1)+(b-1)+b\binom{b}{2}\right)q + \left((b^2-1)(b-1)-b\binom{b}{2}\right)q^2\right)B_b(q) \\
(1-q)^3B_b(3,q) &= \left(1+\left(\tfrac{1}{2}(b-1)(b^2+2b+4)\right)q + \left(\tfrac{1}{2}(b-1)(b^2-2)\right)q^2\right)B_b(q) \\
(1-q)^3B_b(3,q) &= \left(1+(\tfrac{1}{2}(b-1)\left(+(b^2+2b+4)q+(b^2-2)q^2\right)\right)B_b(q)
\end{aligned}
$$

□

Lemma 4.6.

$$
\begin{aligned}
\sum_{k=1}^{b^{m-a}}\sum_{j=0}^{k-1}[\![k \geqslant j]\!]p_b(b^{m-a}(n-1)+k-j) &= \sum_{j=1}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j) \ \textit{moreover} \\
&= \sum_{k=1}^{b^{m-a}-1}kp_b(b^{m-a}n-k)
\end{aligned}
$$

Proof.

$$\begin{aligned}
\text{LHS} &= \sum_{k=1}^{b^{m-a}} \sum_{j=0}^{k-1} [\![k \geqslant j]\!] p_b(b^{m-a}(n-1)+k-j) \text{ let } u = k-j \\
&= \sum_{k=1}^{b^{m-a}-1} \sum_{u=1}^{k} [\![u \geqslant 0]\!] p_b(b^{m-a}(n-1)+u) \text{ let } j = u \\
&= \sum_{k=1}^{b^{m-a}-1} \sum_{j=1}^{k} [\![1 \leqslant j \leqslant k]\!] p_b(b^{m-a}(n-1)+j) \\
&= \sum_{k=1}^{b^{m-a}-1} \sum_{j=1}^{b^{m-a}-1} [\![1 \leqslant j \leqslant k]\!] p_b(b^{m-a}(n-1)+j) \\
&= \sum_{j=1}^{b^{m-a}-1} \sum_{k=1}^{b^{m-a}-1} [\![1 \leqslant j \leqslant k]\!] p_b(b^{m-a}(n-1)+j) \\
&= \sum_{j=1}^{b^{m-a}-1} p_b(b^{m-a}(n-1)+j) \sum_{k=1}^{b^{m-a}-1} [\![1 \leqslant j \leqslant k]\!] \\
&= \sum_{j=1}^{b^{m-a}-1} p_b(b^{m-a}(n-1)+j) \sum_{k=j}^{b^{m-a}-1} 1 \\
&= \sum_{j=1}^{b^{m-a}-1} (b^{m-a}-j) p_b(b^{m-a}(n-1)+j)
\end{aligned}$$

Then

$$\begin{aligned}
\text{LHS} &= \sum_{k=1}^{b^{m-a}} \sum_{j=0}^{k-1} [\![k \geqslant j]\!] p_b(b^{m-a}(n-1)+k-j) \\
&= \sum_{j=1}^{b^{m-a}-1} (b^{m-a}-j) p_b(b^{m-a}(n-1)+j) \text{ let } k = b^{m-a}-j \\
&= \sum_{k=1}^{b^{m-a}-1} k p_b(b^{m-a}(n-1)+b^{m-a}-k) \\
&= \sum_{k=1}^{b^{m-a}-1} k p_b(b^{m-a}n-k)
\end{aligned}$$

□

Corollary 4.7.

$$\sum_{j=1}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j) = \left(\binom{b^{m-a}}{2}-\binom{b^{m-a}-b+1}{2}\right)p_b(b^{m-a}(n-1)) + [\![m>a+1]\!]\sum_{j=b}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j)$$

Proof. Splitting the sum reveals

$$\sum_{j=1}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j) = \sum_{j=1}^{b-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j) + [\![m>a+1]\!]\sum_{j=b}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j)$$

Then the first term becomes

$$\begin{aligned}
\text{LHS} &= \sum_{j=1}^{b-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j) \\
&= \sum_{j=1}^{b-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)) \\
&= p_b(b^{m-a}(n-1))\sum_{j=1}^{b-1}(b^{m-a}-j) \\
&= p_b(b^{m-a}(n-1))\sum_{j=b^{m-a}-b+1}^{b^{m-a}-1} j \\
&= \left(\binom{b^{m-a}}{2}-\binom{b^{m-a}-b+1}{2}\right)p_b(b^{m-a}(n-1))
\end{aligned}$$

Thus

$$\sum_{j=1}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j) = \left(\binom{b^{m-a}}{2}-\binom{b^{m-a}-b+1}{2}\right)p_b(b^{m-a}(n-1)) + [\![m>a+1]\!]\sum_{j=b}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j)$$

□

Theorem 4.8.

$$\begin{aligned}\sum_{k=1}^{b^{m-a}-1} p_b(b^{m-a+1}(n-1)+kb) &= (b^{m-a}-1)p_b(b^{m-a+1}(n-1)) + \sum_{k=1}^{b^{m-a}} kp_b(b^{m-a}n-k) \\ &= (b^{m-a}-1)p_b(b^{m-a+1}(n-1)) \\ &\quad + \left(\binom{b^{m-a}}{2} - \binom{b^{m-a}-b+1}{2}\right) p_b(b^{m-a}(n-1)) \\ &\quad + [\![m > a+1]\!] \sum_{j=b}^{b^{m-a}-1} (b^{m-a}-j)p_b(b^{m-a}(n-1)+j)\end{aligned}$$

Proof.

$$\begin{aligned}\text{LHS} &= \sum_{k=1}^{b^{m-a}-1} p_b(b^{m-a+1}(n-1)+kb) \\ &= \sum_{k=1}^{b^{m-a}-1} \big(p_b(b^{m-a+1}(n-1)+(k-1)b) + p_b(b^{m-a}(n-1)+k)\big) \\ &= \sum_{k=1}^{b^{m-a}-1} \big([\![k>a]\!]p_b(b^{m-a+1}(n-1)+(k-2)b) + p_b(b^{m-a}(n-1)+k-1) + p_b(b^{m-a}(n-1)+k)\big) \\ &= \sum_{k=1}^{b^{m-a}-1} \left(p_b(b^{m-a+1}(n-1)+(k-2)) + \sum_{j=0}^{1}[\![k \geqslant j]\!]p_b(b^{m-a}(n-1)+k-j)\right) \\ &\dots \\ &= \sum_{k=1}^{b^{m-a}-1} \left(p_b(b^{m-a+1}(n-1)) + \sum_{j=0}^{k-1}[\![k \geqslant j]\!]p_b(b^{m-a}(n-1)+k-j)\right) \\ &= (b^{m-a}-1)p_b(b^{m-a+1}(n-1)) + \sum_{k=1}^{b^{m-a}} \sum_{j=0}^{k-1}[\![k \geqslant j]\!]p_b(b^{m-a}(n-1)+k-j)\end{aligned}$$

By Lemma 4.6

$$\sum_{k=1}^{b^{m-a}} \sum_{j=0}^{k-1}[\![k \geqslant j]\!]p_b(b^{m-a}(n-1)+k-j) = \sum_{j=1}^{b^{m-a}-1} (b^{m-a}-j)p_b(b^{m-a}(n-1)+j)$$

Finally by Corollary 4.7:

$$\sum_{j=1}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j) = \left(\binom{b^{m-a}}{2}-\binom{b^{m-a}-b+1}{2}\right)p_b(b^{m-a}(n-1)) + [\![m>a+1]\!]\sum_{j=b}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j)$$

Therefore

$$\begin{aligned}\sum_{k=1}^{b^{m-a}-1}p_b(b^{m-a+1}(n-1)+kb) &= (b^{m-a}-1)p_b(b^{m-a+1}(n-1)) + \sum_{k=1}^{b^{m-a}}kp_b(b^{m-a}n-k)\\ &= (b^{m-a}-1)p_b(b^{m-a+1}(n-1))\\ &\quad+\left(\binom{b^{m-a}}{2}-\binom{b^{m-a}-b+1}{2}\right)p_b(b^{m-a}(n-1))\\ &\quad+[\![m>a+1]\!]\sum_{j=b}^{b^{m-a}-1}(b^{m-a}-j)p_b(b^{m-a}(n-1)+j)\end{aligned}$$

□

The goal now is to evaluate the expression

$$\begin{aligned}\text{LHS} &= p_b(b^m n)\\ &= F^{b^{m-1}}(p_b(b^m n))\\ &= p_b(b^m n - b^m) + \sum_{k=0}^{b^{m-1}-1}p_b(b^{m-1}n-k)\\ &= p_b(b^m(n-1)) + \sum_{k=0}^{b^{m-1}-1}p_b(b^{m-1}n-k)\\ &= p_b(b^m(n-1)) + \sum_{k=1}^{b^{m-1}}p_b(b^{m-1}(n-1)+k)\\ &= p_b(b^m(n-1)) + p_b(b^{m-1}n) + \sum_{k=1}^{b-1}p_b(b^{m-1}(n-1)+k) + [\![m>2]\!]\sum_{k=b}^{b^{m-1}-1}p_b(b^{m-1}(n-1)+k)\\ &= p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) + [\![m>2]\!]\sum_{k=b}^{b^{m-1}-1}p_b(b^{m-1}(n-1)+k)\end{aligned}$$

Note the last term is 0 if $m \leqslant 2$ thus the case $m=2$ is finished here.

The last term then becomes:

$$
\begin{aligned}
\text{LHS} &= \sum_{k=b}^{b^{m-1}-1} p_b(b^{m-1}(n-1)+k) \\
&= \sum_{k=b}^{b^{m-1}-1} p_b(b^{m-1}(n-1)+k) \\
&= \sum_{u=1}^{b^{m-2}-1} \sum_{v=0}^{b-1} p_b(b^{m-1}(n-1)+ub+v) \\
&= \sum_{u=1}^{b^{m-2}-1} \sum_{v=0}^{b-1} p_b(b^{m-1}(n-1)+ub) \\
&= b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1)+ub) \\
&= b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1)+ub)
\end{aligned}
$$

The expression of interest is now:

$$
\begin{aligned}
p_b(b^m n) = p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
+ [\![m > 2]\!] b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1)+ub)
\end{aligned}
$$

Theorem 4.8 can be applied with $a = 2$ to the last term. Then

$$
\begin{aligned}
\text{LHS} &= \sum_{k=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1)+kb) \\
&= (b^{m-2}-1)p_b(b^{m-1}(n-1)) \\
&\quad + \left(\binom{b^{m-2}}{2} - \binom{b^{m-2}-b+1}{2}\right) p_b(b^{m-2}(n-1)) \\
&\quad + [\![m > 3]\!] \sum_{j=b}^{b^{m-2}-1} (b^{m-2}-j)p_b(b^{m-2}(n-1)+j)
\end{aligned}
$$

Note the last term is 0 if $m \leqslant 3$. Thus the case $m = 3$ is finished here (compare Theorem 4.5).

The expression of interest is now:

$$
\begin{aligned}
p_b(b^m n) &= p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
&\quad + [\![m>2]\!] b \sum_{u=1}^{b^{m-2}-1} p_b(b^{m-1}(n-1)+ub) \\
&= p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
&\quad + [\![m>2]\!] b(b^{m-2}-1)p_b(b^{m-1}(n-1)) \\
&\quad + [\![m>2]\!] b\left(\binom{b^{m-2}}{2} - \binom{b^{m-2}-b+1}{2}\right) p_b(b^{m-2}(n-1)) \\
&\quad + [\![m>3]\!] b \sum_{j=b}^{b^{m-2}-1} (b^{m-2}-j)p_b(b^{m-2}(n-1)+j)
\end{aligned}
$$

Again, the action is in the last term:

$$
\begin{aligned}
\text{LHS} &= \sum_{j=b}^{b^{m-2}-1} (b^{m-2}-j)p_b(b^{m-2}(n-1)+j) \\
&= \sum_{u=1}^{b^{m-3}-1} \sum_{v=0}^{b-1} (b^{m-2}-ub-v)p_b(b^{m-2}(n-1)+ub+v) \\
&= \sum_{u=1}^{b^{m-3}-1} \sum_{v=0}^{b-1} (b^{m-2}-ub-v)p_b(b^{m-2}(n-1)+ub) \\
&= \sum_{u=1}^{b^{m-3}-1} p_b(b^{m-2}(n-1)+ub) \sum_{v=0}^{b-1} (b^{m-2}-ub-v) \\
&= \sum_{u=1}^{b^{m-3}-1} p_b(b^{m-2}(n-1)+ub) \left((b^{m-2}-ub)b - \sum_{v=0}^{b-1} v \right) \\
&= \sum_{u=1}^{b^{m-3}-1} p_b(b^{m-2}(n-1)+ub) \left(b^2(b^{m-3}-u) - \binom{b}{2} \right) \\
&= b^2 \sum_{u=1}^{b^{m-3}-1} (b^{m-3}-u)p_b(b^{m-2}(n-1)+ub) - \binom{b}{2} \sum_{u=1}^{b^{m-3}-1} p_b(b^{m-2}(n-1)+ub)
\end{aligned}
$$

Theorem 4.8 can be applied with $a = 3$ to the second term. Then

$$
\begin{aligned}
\text{LHS} &= \sum_{j=b}^{b^{m-2}-1} (b^{m-2}-j)p_b(b^{m-2}(n-1)+j) \\
&= b^2 \sum_{u=1}^{b^{m-3}-1} (b^{m-3}-u)p_b(b^{m-2}(n-1)+ub) - \binom{b}{2} \sum_{u=1}^{b^{m-3}-1} p_b(b^{m-2}(n-1)+ub) \\
&= b^2 \sum_{u=1}^{b^{m-3}-1} (b^{m-3}-u)p_b(b^{m-2}(n-1)+ub) \\
&\quad - \binom{b}{2}(b^{m-3}-1)p_b(b^{m-2}(n-1)) \\
&\quad - \binom{b}{2}\left(\binom{b^{m-3}}{2} - \binom{b^{m-3}-b+1}{2}\right) p_b(b^{m-3}n(n-1)) \\
&\quad - \llbracket m>4 \rrbracket \binom{b}{2} \sum_{j=b}^{b^{m-3}-1} (b^{m-3}-j)p_b(b^{m-3}(n-1)+j)
\end{aligned}
$$

The expression of interest is now:

$$
\begin{aligned}
p_b(b^m n) &= p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
&\quad + \llbracket m>2 \rrbracket b(b^{m-2}-1)p_b(b^{m-1}(n-1)) \\
&\quad + \llbracket m>2 \rrbracket b\left(\binom{b^{m-2}}{2} - \binom{b^{m-2}-b+1}{2}\right) p_b(b^{m-2}(n-1)) \\
&\quad + \llbracket m>3 \rrbracket b \sum_{j=b}^{b^{m-2}-1} (b^{m-2}-j)p_b(b^{m-2}(n-1)+j) \\
&= p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
&\quad + \llbracket m>2 \rrbracket b(b^{m-2}-1)p_b(b^{m-1}(n-1)) \\
&\quad + \llbracket m>2 \rrbracket b\left(\binom{b^{m-2}}{2} - \binom{b^{m-2}-b+1}{2}\right) p_b(b^{m-2}(n-1)) \\
&\quad + \llbracket m>3 \rrbracket b^3 \sum_{u=1}^{b^{m-3}-1} (b^{m-3}-u)p_b(b^{m-2}(n-1)+ub) \\
&\quad - \llbracket m>3 \rrbracket b\binom{b}{2}(b^{m-3}-1)p_b(b^{m-2}n(n-1)) \\
&\quad - \llbracket m>3 \rrbracket b\binom{b}{2}\left(\binom{b^{m-3}}{2} - \binom{b^{m-3}-b+1}{2}\right) p_b(b^{m-3}n(n-1)) \\
&\quad - \llbracket m>4 \rrbracket b\binom{b}{2} \sum_{j=b}^{b^{m-3}-1} (b^{m-3}-j)p_b(b^{m-3}(n-1)+j)
\end{aligned}
$$

known typos unfixed at or past this point involving the b^3 term above

Now consider the sum

$$\begin{aligned}
\text{LHS} &= \sum_{k=1}^{b^{m-3}-1} (b^{m-3}-k)p_b(b^{m-2}(n-1)+kb) \\
&= \sum_{k=1}^{b^{m-3}-1} (b^{m-3}-k)\big(p_b(b^{m-2}(n-1)+(k-1)b)+p_b(b^{m-3}(n-1)+k)\big) \\
&= \sum_{k=1}^{b^{m-3}-1} (b^{m-3}-k)\Big([\![k \geqslant 2]\!]\big(p_b(b^{m-2}(n-1)+(k-2)b)+p_b(b^{m-3}(n-1)+k-1)\big) \\
&\qquad + p_b(b^{m-3}(n-1)+k)\Big) \\
&= \sum_{k=1}^{b^{m-3}-1} (b^{m-3}-k)\left([\![k \geqslant 2]\!]p_b(b^{m-2}(n-1)+(k-2)b)+\sum_{j=0}^{1} p_b(b^{m-3}(n-1)+k-j)\right) \\
&= \sum_{k=1}^{b^{m-3}-1} (b^{m-3}-k)\left(p_b(b^{m-2}(n-1))+\sum_{j=0}^{k-1} p_b(b^{m-3}(n-1)+k-j)\right) \\
&= p_b(b^{m-2}(n-1))\sum_{k=1}^{b^{m-3}-1}(b^{m-3}-k) \text{ let } j=b^{m-3}-k \text{ in this sum} \\
&\qquad + \sum_{k=1}^{b^{m-3}-1}(b^{m-3}-k)\sum_{j=0}^{k-1} p_b(b^{m-3}(n-1)+k-j) \\
&= p_b(b^{m-2}(n-1))\sum_{j=1}^{b^{m-3}-1} j + \sum_{k=1}^{b^{m-3}-1}(b^{m-3}-k)\sum_{j=0}^{k-1} p_b(b^{m-3}(n-1)+k-j) \\
&= \binom{b^{m-3}}{2}p_b(b^{m-2}(n-1)) + \sum_{k=1}^{b^{m-3}-1}(b^{m-3}-k)\sum_{j=0}^{k-1} p_b(b^{m-3}(n-1)+k-j) \text{ let } u=k-j \\
&= \binom{b^{m-3}}{2}p_b(b^{m-2}(n-1)) + \sum_{k=1}^{b^{m-3}-1}\sum_{u=1}^{k}(b^{m-3}-k)p_b(b^{m-3}(n-1)+u) \text{ let } u=j \\
&= \binom{b^{m-3}}{2}p_b(b^{m-2}(n-1)) + \sum_{k=1}^{b^{m-3}-1}\sum_{j=1}^{k}(b^{m-3}-k)p_b(b^{m-3}(n-1)+j)
\end{aligned}$$

The expression of interest is now:

$$
\begin{aligned}
p_b(b^m n) = {} & p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
& + [\![m>2]\!] b(b^{m-2}-1)p_b(b^{m-1}(n-1)) \\
& + [\![m>2]\!] b\left(\binom{b^{m-2}}{2} - \binom{b^{m-2}-b+1}{2}\right) p_b(b^{m-2}(n-1)) \\
& + [\![m>3]\!] b^3 \sum_{u=1}^{b^{m-3}-1} (b^{m-3}-u)p_b(b^{m-2}(n-1)+ub) \\
& - [\![m>3]\!] b\binom{b}{2}(b^{m-3}-1)p_b(b^{m-2}n(n-1)) \\
& - [\![m>3]\!] b\binom{b}{2}\left(\binom{b^{m-3}}{2} - \binom{b^{m-3}-b+1}{2}\right) p_b(b^{m-3}n(n-1)) \\
& - [\![m>4]\!] b\binom{b}{2} \sum_{j=b}^{b^{m-3}-1} (b^{m-3}-j)p_b(b^{m-3}(n-1)+j) \\
= {} & p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
& + [\![m>2]\!] b(b^{m-2}-1)p_b(b^{m-1}(n-1)) \\
& + [\![m>2]\!] b\left(\binom{b^{m-2}}{2} - \binom{b^{m-2}-b+1}{2}\right) p_b(b^{m-2}(n-1)) \\
& + [\![m>3]\!] b^3 \binom{b^{m-3}}{2} p_b(b^{m-2}(n-1)) \\
& + [\![m>3]\!] b^3 \sum_{k=1}^{b^{m-3}-1} \sum_{j=1}^{k} (b^{m-3}-k)p_b(b^{m-3}(n-1)+j) \\
& - [\![m>3]\!] b\binom{b}{2}(b^{m-3}-1)p_b(b^{m-2}n(n-1)) \\
& - [\![m>3]\!] b\binom{b}{2}\left(\binom{b^{m-3}}{2} - \binom{b^{m-3}-b+1}{2}\right) p_b(b^{m-3}n(n-1)) \\
& - [\![m>4]\!] b\binom{b}{2} \sum_{j=b}^{b^{m-3}-1} (b^{m-3}-j)p_b(b^{m-3}(n-1)+j)
\end{aligned}
$$

Now consider the sum

$$
\begin{aligned}
\text{LHS} &= \sum_{k=1}^{b^{m-3}-1} \sum_{j=1}^{k} (b^{m-3}-k) p_b(b^{m-3}(n-1)+j) \\
&= \sum_{k=1}^{b^{m-3}-1} \sum_{j=1}^{k} [\![1 \leqslant j \leqslant k]\!] (b^{m-3}-k) p_b(b^{m-3}(n-1)+j) \\
&= \sum_{k=1}^{b^{m-3}-1} \sum_{j=1}^{b^{m-3}-1} [\![1 \leqslant j \leqslant k]\!] (b^{m-3}-k) p_b(b^{m-3}(n-1)+j) \\
&= \sum_{j=1}^{b^{m-3}-1} \sum_{k=1}^{b^{m-3}-1} [\![1 \leqslant j \leqslant k]\!] (b^{m-3}-k) p_b(b^{m-3}(n-1)+j) \\
&= \sum_{j=1}^{b^{m-3}-1} p_b(b^{m-3}(n-1)+j) \sum_{k=1}^{b^{m-3}-1} [\![1 \leqslant j \leqslant k]\!] (b^{m-3}-k) \\
&= \sum_{j=1}^{b^{m-3}-1} p_b(b^{m-3}(n-1)+j) \sum_{k=j}^{b^{m-3}-1} (b^{m-3}-k) \text{ let } u = b^{m-3}-k \\
&= \sum_{j=1}^{b^{m-3}-1} p_b(b^{m-3}(n-1)+j) \sum_{u=1}^{b^{m-3}-j} u \\
&= \sum_{j=1}^{b^{m-3}-1} p_b(b^{m-3}(n-1)+j) \binom{b^{m-3}-j+1}{2} \\
&= \sum_{j=1}^{b^{m-3}-1} \binom{b^{m-3}-j+1}{2} p_b(b^{m-3}(n-1)+j) \\
&= \sum_{j=1}^{b-1} \binom{b^{m-3}-j+1}{2} p_b(b^{m-3}(n-1)+j) \\
&\qquad + [\![m > 4]\!] \sum_{j=b}^{b^{m-3}-1} \binom{b^{m-3}-j+1}{2} p_b(b^{m-3}(n-1)+j)
\end{aligned}
$$

Focusing on the first sum

$$\begin{aligned}
\sum_{j=1}^{b-1}\binom{b^{m-3}-j+1}{2}p_b(b^{m-3}(n-1)+j) &= \sum_{j=1}^{b-1}\binom{b^{m-3}-j+1}{2}p_b(b^{m-3}(n-1)) \\
&= p_b(b^{m-3}(n-1))\sum_{j=1}^{b-1}\binom{b^{m-3}-j+1}{2} \text{ let } k = b^{m-3}-j+1 \\
&= p_b(b^{m-3}(n-1))\sum_{k=b^{m-3}-b+2}^{b^{m-3}}\binom{k}{2} \\
&= p_b(b^{m-3}(n-1))\left(\binom{b^{m-3}+1}{3}-\binom{b^{m-3}-b+2}{3}\right) \\
&= \left(\binom{b^{m-3}+1}{3}-\binom{b^{m-3}-b+2}{3}\right)p_b(b^{m-3}(n-1))
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{k=1}^{b^{m-3}-1}\sum_{j=1}^{k}(b^{m-3}-k)p_b(b^{m-3}(n-1)+j) &= \left(\binom{b^{m-3}+1}{3}-\binom{b^{m-3}-b+2}{3}\right)p_b(b^{m-3}(n-1)) \\
&\quad + [\![m>4]\!]\sum_{j=b}^{b^{m-3}-1}\binom{b^{m-3}-j+1}{2}p_b(b^{m-3}(n-1)+j)
\end{aligned}$$

The expression of interest is now:

$$
\begin{aligned}
p_b(b^m n) = {} & p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
& + [\![m>2]\!] b(b^{m-2}-1)p_b(b^{m-1}(n-1)) \\
& + [\![m>2]\!] b\left(\binom{b^{m-2}}{2} - \binom{b^{m-2}-b+1}{2}\right) p_b(b^{m-2}(n-1)) \\
& + [\![m>3]\!] b^3\binom{b^{m-3}}{2} p_b(b^{m-2}(n-1)) \\
& + [\![m>3]\!] b^3 \sum_{k=1}^{b^{m-3}-1}\sum_{j=1}^{k}(b^{m-3}-k)p_b(b^{m-3}(n-1)+j) \\
& - [\![m>3]\!] b\binom{b}{2}(b^{m-3}-1)p_b(b^{m-2}n(n-1)) \\
& - [\![m>3]\!] b\binom{b}{2}\left(\binom{b^{m-3}}{2} - \binom{b^{m-3}-b+1}{2}\right) p_b(b^{m-3}n(n-1)) \\
& - [\![m>4]\!] b\binom{b}{2}\sum_{j=b}^{b^{m-3}-1}(b^{m-3}-j)p_b(b^{m-3}(n-1)+j) \\
= {} & p_b(b^m(n-1)) + p_b(b^{m-1}n) + (b-1)p_b(b^{m-1}(n-1)) \\
& + [\![m>2]\!] b(b^{m-2}-1)p_b(b^{m-1}(n-1)) \\
& + [\![m>2]\!] b\left(\binom{b^{m-2}}{2} - \binom{b^{m-2}-b+1}{2}\right) p_b(b^{m-2}(n-1)) \\
& + [\![m>3]\!] b^3\binom{b^{m-3}}{2} p_b(b^{m-2}(n-1)) \\
& + [\![m>3]\!] b^3\left(\binom{b^{m-3}+1}{3} - \binom{b^{m-3}-b+2}{3} + X\right) p_b(b^{m-3}(n-1)) \\
& - [\![m>3]\!] b\binom{b}{2}(b^{m-3}-1)p_b(b^{m-2}n(n-1)) \\
& - [\![m>3]\!] b\binom{b}{2}\left(\binom{b^{m-3}}{2} - \binom{b^{m-3}-b+1}{2}\right) p_b(b^{m-3}n(n-1)) \\
& - [\![m>4]\!] b\binom{b}{2}\sum_{j=b}^{b^{m-3}-1}(b^{m-3}-j)p_b(b^{m-3}(n-1)+j) \\
& + [\![m>4]\!] b^3\sum_{j=b}^{b^{m-3}-1}\binom{b^{m-3}-j+1}{2}p_b(b^{m-3}(n-1)+j)
\end{aligned}
$$

And the case $m=4$ is done here, except for the mystery term of $X=\binom{b-1}{3}/2$. This term probably represents a typo in the (currently 8 page) argument.

m	$b=2$	$b=3$	$b=4$	$b=5$
3	$1+6q+q^2$	$1+19q+7q^2$	$1+42q+21q^2$	$1+78q+46q^2$
4	$1+31q+31q^2+q^3$	$1+234q+447q^2+47q^3$	$1+1081q+2635q^2+379q^3$	$1+3072q+10218q^2+1704q^3$
5	$1+196q+630q^2+196q^2+q^4$	$1+5822q+33504q^2+19040q^3+682q^4$		